Math 171

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1 September 26

1.1 The Set of Real Numbers

Def. The set $\mathbb{R}$of **real numbers** is a set satisfying the following 13 axioms.

1. $\exists$ a binary operation such that for any $x, y \in \mathbb{R}$, $x + y \in \mathbb{R}$.
2. Addition is associative: $x + (y + z) = (x + y) + z$.
3. Addition is commutative: $x + y = y + x$.
4. An additive identity exists: $x + 0 = x \ \forall x \in \mathbb{R}$.

   Remark: It's simple to prove this additive identity is unique. Assume there is a second additive identity 0'. Then $0 + 0' = 0 = 0'$. Therefore $0 = 0'$, and they're the same number (therefore unique zero element).
5. An additive inverse exists: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 0$.

Remark: We can use 1. - 4. to prove it is unique.

6. There is a binary operation $*$ such that $\forall x, y \in \mathbb{R}, x * y \in \mathbb{R}$ (typically denoted as xy).
7. Multiplication is associative: x(yz) = (xy)z.
8. Multiplication is commutative: xy = yx.
9. A multiplication identity exists: $\exists 1 \in \mathbb{R}, 1 \neq 0$ such that $1 * x = x$.

   Remark: The set $\{0\}$ would satisfy 1-8. Axiom 9 is the first one which requires the set be composed of additional elements.

10. A multiplicative inverse exists for non-zero real numbers: $\forall x \in \mathbb{R}, x \neq 0, \exists y \in \mathbb{R}$ such that $xy = 1$.

    Remark: We can prove this is unique with 1-9.

    Remark: Cannot have an inverse of 0: it would violate 9 since by definition $x * 0 = 0$ and $1 \neq 0$.

11. Multiplication distributes over addition: $x(y + z) = xy + xz$.
12. Order Axiom: $\exists P \subset \mathbb{R}$ callex the set of "positive real numbers" such that:
    (a) $x, y \in P \Rightarrow x + y \in P$.
    (b) If $x \in \mathbb{R}$, then exactly one of the following is true:
        i. $x \in P$.
        ii. $-x \in P$
        iii. $x = 0$

    Notation: (a) $x > y$ means $x - y \in P$
    (b) $x < y$ means $y - x \in P$
    (c) $x \geq y$ means $x - y \in P$ or $x = y$
    (d) $x \leq y$ means $y - x \in P$ or $x = y$

    Remark: This axiom offers a notion of "ordering" which allows us to compare two numbers in the sense of "bigger" or "smaller".

13. Completeness Axiom (or Least Upper Bound axiom): Any non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

    Upper Bound: We say $A \subset \mathbb{R}$ is bounded above if $\exists a \in \mathbb{R}$ such that $\forall x \in A$, $x \leq a$. Any such a is called an upper bound of A.

    Least Upper Bound: An upper bound a of A is called a least upper bound if for any other upper bound b, $a \leq b$.

Remark: $\exists$ subset of rational numbers with no least upper bound (i.e. the set $\{x \in \mathbb{Q} | x^2 < 2\}$, since we can always find a number that's a little closer to $\sqrt{2}$ in the set of rational numbers than the number we assume to be the supremum.

Remark: The completeness axiom separates the Real numbers from the rational numbers.

1.2 Dedekind Cuts

Dedekind Cuts is a method to define the real numbers from $\mathbb{Q}$ and $\mathbb{N}$. **Def.**: A Dedekind Cut of $\mathbb{Q}$ is a set $r \in \mathbb{Q}$ such that:

- $r \neq \{\}$.
- $r \neq \mathbb{Q}$.
- If $y \in r, x \in \mathbb{Q}, x < y$, then $x \in r$ (i.e. all rationals less than y are in the "Dedekind cut" r.
- $\nexists x \in r$ such that $x \geq y \; \forall y \in r$ (i.e. no supremum or upper bound of r).

Ex.

- r = $\{x \in \mathbb{Q} | x < 2\}$ is a dedekind cut.
- $s = \{x \in \mathbb{Q} | x^2 < 2\} \cup \{x \in \mathbb{Q} | x < 0\}$ is the union of two dedekind cuts.

Remark Dedekind defined $\mathbb{R}$to be the set of all Dedekind cuts of $\mathbb{Q}$. Each set of Dedekind Cuts then represents a single Real Number (i.e. $\sqrt{2} = \{x \in \mathbb{Q} | x^2 < 2\}$). You can then define axioms on this set.

Theorem 1.1 *$\sqrt{2}$ is irrational: $\nexists x \in \mathbb{Q}$ s.t. $x^2 = 2$.*

Proof. [by contradiction]. Suppose not. Then $\exists x \in \mathbb{Q}$ s.t. $x^2 = 2$. So we can write $x = \frac{p}{q}$ for some coprime integers (i.e. integers that share no common factors). Expanding on the last point, we can write any rational number as a fraction of coprime integers $p, q \in \mathbb{Z}$ (co-prime: cancel out the common factors). Then $x^2 = \frac{p^2}{q^2} = 2$. And p^2 since 2 is even, therefore p is even, therefore $p = 2n$ for some integer n. And $2q^2 = p^2 = 4n^2 \Rightarrow q^2 = 2n^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even. Therefore, we arrive at a contradiction since p and q are co-prime.

Def. If $x \in \mathbb{R}$, the **absolute value** of x is defined as

1. $|x| = x$ if $x > 0$.
2. $|x| = -x$ if $x < 0$.
3. $|x| = 0$ if $x = 0$.

.

Theorem 1.2 *Triangle Inequality: $|x + y| \leq |x| + |y|$*

2 September 28

2.1 Dedekind Cuts (cont.)

Suppose r is a Dedekind Cut. We say:

- We say $r < s$ (one Dedekind cut is less than another) if r is a proper subset of s (i.e. s contains all elements of r and at least one more element that is not in r): $r \subsetneq s$.
- We can define 0 as: $\{x \in \mathbb{Q} | x < 0\}$.
- We can define -r as: -r := $\{x | \exists y > x, -y \in r\}$ or -r := 0 - r = $\{x - y | x \in 0, y \in \mathbb{Q} \setminus r\}$.
- A is a set of Dedekind cuts. The least upper bound of A is just $\cup_{r \in A} r$.

2.2 Supremum and Infinum

Def. Archimedean Property of Real Numbers. If n is a positive integer, we will denote by n the real number 1 + 1 + 1 + ... + 1 (n-times).

- This represents an embedding of natural numbers within $\mathbb{R}$.
- We can do this with $\mathbb{Q}$as well.
- $\Rightarrow \mathbb{N} \subset \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$.

Theorem 2.1 *Given any $\epsilon > 0$, $\exists n$ (i.e. a positive integer) such that $n * \epsilon > 1$ (i.e. add ϵ n times).*

Proof: Suppose y = sup(S) $\in \mathbb{R}$ and note $y - \epsilon < y$. Then $y - \epsilon$ is not an upper bound. Therefore, $\exists n * \epsilon > y - \epsilon \Rightarrow n * \epsilon + \epsilon = \epsilon(n+1) > y \Rightarrow$ contradiction.

Corollary 2.1.1 *Let x be a non-negative (i.e. $x \geq 0$) real number. If $x < \epsilon, \forall \epsilon > 0$, then $x = 0$.*

Proof: Since $x < \epsilon \forall \epsilon > 0$, we have $x < \frac{1}{n} \forall n \in \mathbb{N}$. Therefore, $n * x < 1 \ \forall n$. Therefore $x = 0$ since $x \not> 0$ (by the Archimedean Property). Expanding on the last point, Given any $\epsilon > 0$, $\exists n$ such that $n * \epsilon > 1$, and therefore $x \not> 0$ and $x = 0$.

2.3 Sequence of Real Numbers

An infinite list of real numbers $x_1, x_2, x_3, ...$ is called a sequence.
Def.

- Let $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers. We say that $\{x_n\}_{n=1}^{\infty}$converges to a limit x if:

  1. $\forall \epsilon > 0$, $\exists N \geq 1$ such that $\forall n \geq N$, $|x_n - x| < \epsilon$.

2. x is denoted by $\lim\limits_{n \to \inf} x_n$.
3. For a limit x, and any difference ϵ, $\exists$ some $N \geq 1$ such that for all numbers greater than N, the difference of further elements in this sequence is $< \epsilon$.

Ex. $\lim\limits_{n \to \inf} \frac{n}{n+1} = 1$.
$x_n = \frac{n}{n+1}$ and $|x_n - 1| = |\frac{n}{n+1} - 1| = |\frac{-1}{n+1}| = \frac{1}{n+1}$. Take any $\epsilon > 0$, by the Archimedean property, $\exists N$ such that $N * \epsilon > 1 \Rightarrow \frac{1}{N} < \epsilon$. Thus $\forall n \geq N$, $|x_n - 1| = \frac{1}{n+1} < \frac{1}{N} < \epsilon$.

Theorem 2.2 *If x and x' are both limits of the same sequence $\{x_n\}_{n=1}^{\infty}$, then x = x'.*

Proof: Take $\epsilon > 0 \Rightarrow \exists N_1$ s.t. $\forall n \geq N_1, |x_n - x| < \frac{\epsilon}{2}$ and $\exists N_2$ s.t. $\forall n \geq N_2, |x_n - x'| < \frac{\epsilon}{2}$ [definition of the limit of the sequence]. Then we can take any $n \geq \max\{N_1, N_2\}$ such that $|x_n - x| < \frac{\epsilon}{2}$ and $|x_n - x'| < \frac{\epsilon}{2}$. Thus $|x - x'| \leq |x - x_n| + |x_n - x'| < \epsilon$ [Reverse triangle inequality]. So $|x - x'| < \epsilon \ \forall \epsilon > 0$, but $|x - x'| \geq 0$, so $|x - x'| = 0$ and thus x' = x.

Theorem 2.3 (Sandwiching Principle/Squeeze Theorem) *Let $a_n \leq b_n \leq c_n \forall n$. If $\lim\limits_{n \to \inf} a_n = \lim\limits_{n \to \inf} c_n$ and both limits exist, then $\lim\limits_{n \to \inf} a_n = \lim\limits_{n \to \inf} b_n = \lim\limits_{n \to \inf} c_n$.*

Proof: Let $L = \lim\limits_{n \to \inf} a_n = \lim\limits_{n \to \inf} c_n$. Take any $\epsilon > 0 \Rightarrow \exists N_1$ such that $\forall n \geq N_1$, $|a_n - L| < \epsilon$. Similarly, $\exists N_2$ such that $\forall n \geq N_2, |c_n - L| < \epsilon$. Let N = $\max\{N_1, N_2\}$, take any $n \geq N_1$, then $|a_n - L| < \epsilon$, $|c_n - L| < \epsilon$. Therefore, $L - \epsilon < a_n < L + \epsilon$ and $L - \epsilon < c_n < L + \epsilon$. But we know $a_n \leq b_n \leq c_n$ $\Rightarrow L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon \Rightarrow L - \epsilon < b_n < L + \epsilon \Rightarrow |b_n - L| < \epsilon$ and therefore $\lim\limits_{n \to \inf} b_n = L$.

Def. [Bounded Sequence]. $\{x_n\}_{n=1}^{\infty}$is called "bounded above" if $\exists M \in \mathbb{R}$ such that $\forall n, x_n \leq M$.

- Similar for bounded below.
- A sequence is "bounded" if it is both bounded above and below.

3 September 30

3.1 Bounds of Sequences

Theorem 3.1 *Any convergent sequence is bounded (bounded above and below).*

Proof: Let $\{x_n\}_{n=1}^{\infty}$be a sequence converging to a limit x. Then $\exists N$ such that $\forall n \geq N$, $|x_n - x| < 1$ (typical criterion for convergence where we choose $\epsilon = 1$).

Then $\forall n \geq N$, $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$ (middle equality via triangle inequality). If $n < N$, $x_n \leq \max\{|x_1|, |x_2|, ..., |x_{N-1}|$. Thus $\forall n$, $|x_n| \leq \max\{|x_1|, |x_2|, ..., |x_{N-1}|, 1 + |x|\}$, and we've therefore found a fixed upper and lower bound (since we bound the absolute value of x). Expanding on the last inequalities, we find that x_n is less or equal to than the maximum of the absolute value of any element in the sequence up to the Nth element (including the nth element). And after the Nth element, we know $|x_n| < 1 + |x|$. Therefore we can combine these two partitions of the sequence to find the upper and lower bound (i.e. a fixed bound on the $|x_n|$).
Def.[Monotone Sequences]

- A sequence is called increasing if $x_n \leq x_{n+1} \forall n$.
- A sequence is called decreasing if $x_n \geq x_{n+1} \forall n$.
- A sequence is either increasing or decreasing.

Theorem 3.2 *Any increasing sequence $\{x_n\}_{n=1}^{\infty}$that is bounded above converges to the limit which is the $sup_{n \geq 1} x_n$ (i.e. a single number). Similarly, any decreasing sequence that is bounded below converges to $inf_{n \geq 1} x_n$.*

Remark. This is clearly not true for $\mathbb{Q}$: teh sequence approaching $\sqrt{2}$ is increasing (i.e. add a digit) but converges to a number $\sqrt{2}$ that is outside the set of rationals. *Proof:* Assume $\{x_n\}_{n=1}^{\infty}$is an increasing sequence which is bounded above. Then x = $sup_{n \geq 1} x_n$ is defined as the supremum of the sequence (and it exists..). Take $\epsilon > 0$. Then $x - \epsilon < x$ and $x - \epsilon$ is not an upper bound. Therefore, there exists an element in $\{x_n\}_{n=1}^{\infty}$such that $x - \epsilon < x_n$. But since $\{x_n\}_{n=1}^{\infty}$is an increasing sequence, this implies $x - \epsilon < x_n \forall n \geq N$. But $x_n \leq x \forall n$. Therefore, $\forall n \geq N$, $x - \epsilon < x_n \leq x$. Therefore, $|x - x_n| < \epsilon$ and the sequence converges to x. To prove $|x - x_n| < \epsilon$ you need both $x - x_n < \epsilon$ as well as $x_n - x < \epsilon$.

Theorem 3.3 (Bolzano-Weierstrass Theorem) *Any bounded sequence has a subsequence that converges to a limit.*

Def.[Subsequence]: Suppose that $\{x_n\}_{n=1}^{\infty}$is a sequence and $n_1 < n_2 < n_3 < ..., < n$ is a strictly increasing sequence of natural numbers. Then

- $x_{n_1}, x_{n_2}, x_{n_3}, ...$ is called a subsequence of $x_1, x_2, ...$
- $x_{n_2}, x_{n_4}, x_{n_6}, ...$ is a subsequence.
- $x_{n_2}, x_{n_3}, x_{n_5}, ...$ is a subsequence.
- $x_{n_2}, x_{n_1}, x_{n_4}, ...$ is not a subsequence.

Proof: Since x_n is a bounded sequence, $\exists a, b \in \mathbb{R}$ such that $x_n \in [a, b]$ (this notation indicates every x_n is between the interval a and b) $\forall n$. i.e. $[a, b] = \{x | a \leq x \leq b\}$, $(a, b) = \{x | a < x < b\}$, $[a, b) = \{x | a \leq x < b\}$.

Now let $[a, b] = [a, \frac{a+b}{2} \cup [\frac{a+b}{2}, b]$. Then at least one half (although both may) must contain x_n for infinitely many n (since $\{x_n\}_{n=1}^{\infty}$is infinite). Let $[a_1, b_1]$ be one such half (if both halves contain infinitely many x_n, simply choose one). Now let again $[a_1, b_1] = [a_1, \frac{a_1+b_1}{2}] \cup [\frac{a_1+b_1}{2}, b_1]$. Again, at least one half must contain infinitely many x_n's. Let $[a_2, b_2]$ be one such half...

Note: $[a_{n+1}, b_{n+1}] \subset [a_n, b_n] \forall n$. Thus $a_1 \leq a_2 \leq a_3, ...$ and $b_1 \geq b_2 \geq b_3, ...$ are monotone sequences and are bounded inside $[a, b]$. Therefore $\lim a_n$ and $\lim b_n$ exist and are bounded above and below [using any increasing sequence that is bounded above converges to the supremum and any sequence that is bounded below converges to the infinum]! Note, $b_n - a_n = \frac{b-a}{2^n} \Rightarrow b_n - a_n \to 0$. So the limit of both sequences is L: $\lim a_n = \lim b_n = L$ and there's a subsequence of $a_1, a_2, ...$ and $b_1, b_2, b_3, ...$ that exists within x_n.
Claim: $\exists$ a subsequence $x_{n_1}, x_{n_2}, ...$ converging to L. *Proof:* Since $[a_1, b_1]$ contains x_n for infinitely many n, we can find n_1 such that $x_{n_1} \in [a_1, b_1]$. Since $[a_2, b_2]$ contains x_n for infinitely many n, we can find $n_2 > n_1$ such that $x_{n_2} \in [a_2, b_2]$. Continuing this way, we could find a strictly increasing sequence $n_1 < n_2 < n_3, ...$ such that $\forall k, x_{n_k} \in [a_k, b_k]$. Therefore, $a_k \leq x_{n_k} \leq b_k \forall k$. By the sandwiching principle (squeeze theorem), $\lim_{k \to \inf} x_{n_k} = L$.
Def.[Cauchy]: A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$is called "cauchy" if $\forall \epsilon > 0, \exists N$ such that $\forall m, n \geq N, |x_m - x_n| < \epsilon$. This is even stronger than convergence (i.e. after a certain index in the sequence, the absolute value of the difference of any two elements of the sequence is less than an arbitrary value.

Theorem 3.4 *A sequence of real numbers is convergent if and only if it is Cauchy.*

4 October 3

4.1 Cauchy Sequences

Theorem 4.1 *A sequence $\{x_n\}_{n=1}^{\infty}$of real numbers is Cauchy if and only if it is convergent.*

Remark.
Don't even need to know the limit, if you can tell it's cauchy, it's convergent.

Prof:
Convergent $\Rightarrow$ Cauchy
Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$converges to a limit x. Take any $\epsilon > 0$. Find N so large that $\forall n \geq N$, $|x_n - x| < \frac{\epsilon}{2}$ [from the definition of convergence to a limit]. Take any $m, n \geq N$, Then by the triangle inequality $|x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Convergent $\Leftarrow$ Cauchy
Let $\{x_n\}_{n=1}^{\infty}$be a Cauchy sequence.

Step 1: $\{x_n\}_{n=1}^{\infty}$is a bounded sequence.
Proof: $\exists N$ such that $\forall m, n \geq N$, $|x_m - x_n| < 1$. In particular, for all $n \geq N$, $|x_N - x_n| < 1 \Rightarrow |x_n| \leq |x_N - x_n| + |x_N| < 1 + |x_N|$.

We've seen this technique before for bounding a sequence. First bound everything after index N using convergence properties, and then bound everything before N by taking the max of a finite sequence.

But for $n < N$, $|x_n| \leq \max\{|x_1|, ..., |x_{N-1}|\}$. Thus $\forall n$, $|x_n| \leq \max\{|x_1|, ..., |x_{N-1}|, 1 + |x_N|\}$. Therefore, $\{x_n\}_{n=1}^{\infty}$is bounded (both above and below because we find an upper bound for $|x_n|$.

Step 2: By Bolzano-Weierstrass, there exists a subsequence $\{x_n\}_{n=1}^{\infty}$converging to some x. [We don't know what the limit is, but there is some limit.]

Step 3: $\lim\limits_{n \to \infty} x_n = x$ [The whole sequence converges to a limit – not just the subsequence].
Proof: Take any $\epsilon > 0$. Since $x_{n_k} \to x$ as $k \to \infty$, $\exists K$ such that $\forall k \geq K$, $|x_{n_k} - x| < \frac{\epsilon}{2}$. On the other hand, since $\{x_n\}_{n=1}^{\infty}$is Cauchy, $\exists N$ such that $\forall m, n \geq N$, $|x_n - x_m| < \frac{\epsilon}{2}$ [Cauchy property]. Now recall $n_1 < n_2 < n_3 < ...$ is strictly increasing. Take any $n \geq N$. $\exists k$ such that $k \geq K$, and $n_k \geq N$.

We're using Step 2 to find a subsequence of indices k such that x_{n_k} converges to a limit x as well as the property of $\{x_n\}_{n=1}^{\infty}$being Cauchy. If we pick a index that's sufficiently far along...

- Then $|x_{n_k} - x| < \frac{\epsilon}{2}$ since $k \geq K$.
- $|x_n - x_{n_k}| < \frac{\epsilon}{2}$ since n, $n_k \geq N$.

Thus, $|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, and $\{x_n\}_{n=1}^{\infty}$converges to x.

4.2 Convergence to Infinity

Remark: convergence to infinity

- We say that $\lim\limits_{n \to \infty} x_n = \infty$, if for any $x \in \mathbb{R}$, $\exists N$ such that $\forall n \geq N$, $x_n > x$.
- $\lim\limits_{n \to \infty} x_n = -\infty$, if $\forall x \in \mathbb{R}$, $\exists N$ such that $\forall n \geq N$, $x_n < x$.

Theorem 4.2 *Any monotone sequence of real numbers converges to a limit in* $\mathbb{R} \cup \{-\infty, \infty\}$.

Remark: We should try to prove this on our own later on.

4.3 Limsup and Liminf

Def.[Limsup of a sequence]
Let $\{x_n\}_{n=1}^{\infty}$be any sequence of real numbers. For each n, let $b_n = \sup\{a_k|k \geq n\}$ where the supremum is defined to be ∞ if the set $\{a_k|k \geq n\}$ (i.e. the least upperbound on the sequence of all values that come after the current index n) is unbounded above.

Then $b_1 \geq b_2 \geq b_3 \geq b_4, ...$ is a strictly decreasing sequence since at each successive index, we take the supremum of a smaller set of numbers (holds for convergence to infinity as well). We are presented with a decreasing sequence of real numbers, so the limit $\lim\limits_{n\to\infty} b_n$ exists and is in $\mathbb{R} \cup \{-\infty, \infty\}$. This limit is called the "limit superior" of $\{x_n\}_{n=1}^{\infty}$and is denoted by $\limsup\limits_{n\to\infty} a_n$. Any sequence has a lim sup, and it's always well defined (even if the sequence itself does not have a limit).

Def.[Liminf of a sequence]
Let $c_n = \inf\{a_k|k \geq n\}$. The inf is $-\infty$ if the set $\{a_k|k \geq n\}$ is bounded below. Then $c_1 \leq c_2 \leq c_3 \leq ....$ $\liminf\limits_{n\to\infty} a_n := \lim\limits_{n\to\infty} c_n \in \mathbb{R} \cup \{-\infty, \infty\}$.

Ex. $\{-1, 1, -1, 1, -1, 1, ...\}$ $\limsup a_n = 1$ and $\liminf a_n = -1$ [although the limit itself of the sequence does not exist]. Intuitively, the lim sup is the smallest number such that the sequence goes above it finitely many times. Anything smaller than this value, the sequence will go below it infinitely many times. Accordingly, the limsup or liminf are mainly used to prove that limits exist...

Theorem 4.3 *For any sequence $\{x_n\}_{n=1}^{\infty}$, the liminf $\leq$ limsup.*

Proof: Let $b_n = \sup_{k\geq n} a_k$ and $c_n = \inf_{k\geq n} a_k$. Then $c_n \leq b_n$ $\forall n$. Thus, $\liminf a_n = \lim c_n \leq \lim b_n = \limsup a_n$. If liminf = limsup, that's a necessary and sufficient condition for the limit of the sequence to exist.

Theorem 4.4 *If the* $\limsup a_n = \liminf a_n$*, then* $\lim a_n$ *exists and is equal to this number. Conversely, if* $\lim a_n$ *exists, then* $\limsup a_n = \liminf a_n = \lim a_n$.

Proof:
First, suppose that $\limsup a_n = \liminf a_n = L \in \mathbb{R}$.(Your own exercise: prove it for L = $\infty, -\infty$).

Let $b_n = \sup_{k\geq n} a_k$, $c_n = \inf_{k\geq n} a_k$. Then, $c_n \leq a_n \leq b_n$. But $c_n \to L$ and $b_n \to L$. Thus by sandwiching $a_n \to L$. ($b_n = \sup\{a_n, a_{n+1}, a_{n+2}, ...\}$, thus a_n is clearly less than or equal to b_n – same reasoning applies to inf). We could prove the sandwiching principle holds even if limit is ∞ or $-\infty$.

Conversely, suppose that $\lim\limits_{n\to\infty} a_n = L \in \mathbb{R}$.

Take any $\epsilon > 0$, Then $\exists n$ such that $|a_n - L| < \epsilon \ \forall n \geq N$. In particular, $\forall n \geq N$, $a_n < L + \epsilon$. This implies, $b_N = \sup\{a_n | n \geq N\} \leq L + \epsilon$ [$L + \epsilon$ is the upper bound of the set]. $b_1 \geq b_2 \geq ... \to \limsup a_n$ Thus, $\limsup a_n \leq L + \epsilon$.

Note this relation holds for $\epsilon > 0$ and no longer contains N (independent of the index you choose in the sequence): $\limsup a_n \leq L$.

By a similar argument, $\liminf a_n \geq L$. Thus $\limsup a_n \leq \liminf a_n$, but we know that the $\limsup a_n \geq \liminf a_n$, therefore both limits are equal: $\limsup a_n = \liminf a_n$. And moving to the Limit L, $\limsup a_n \leq L \leq \liminf a_n$. Thus $\limsup a_n = \liminf a_n = L$.

To review, if the limsup and liminf both exist and are equal, then then the limit exists and is equal to that number. Conversely, if the limit exists, then the limsup and liminf exists and both converge to that same number. This technique may help us understand is oscillating (i.e. one of liminf or limsup converges but not the other; or they are not equal to each other).

In this proof, we used the fact, we used the following fact to lower the bound on lim sup and lim inf from $L + \epsilon$ to L: $x \leq y + \epsilon \ \forall \epsilon > 0 \Rightarrow x \leq y$.

5 October 5

5.1 Limsup and Liminf

Recall the $\limsup\limits_{n\to\infty} a_n$ is a single value representing what the supremum of successively smaller subsets of $\{a_n\}_{n=1}^{\infty}$ converges to. More formally, define another sequence $\{b_n\}_{n=1}^{\infty}$ such that $b_n = \sup\{a_k | k \geq n\}$. Denote an element of the sequence $b_n = \sup_{k\geq n} a_n$. Then we have the following properties:

- $\sup_{n\geq 1}(a_n + b_n) \leq \sup_{n\geq 1} a_n + \sup_{n\geq 1} b_n$.
  - This is because $a_i + b_i$ can cancel each other out.
  - e.g. {1, 2, 3, 4} with {-1, -2, -3, -4} – $\sup_{n\geq 1} a_n = 4$ and $\sup_{n\geq 1} b_n = -1$, but $\sup_{n\geq 1} a_n + b_n = 0$
- Extending this to the lim sup as $n \to \infty$, $\forall n, \sup_{k\geq n}(a_k + b_k) \leq sup_{k\geq n} a_k + sup_{k\geq n} b_k \Rightarrow \limsup\limits_{n\to\infty} a_n + b_n \leq \limsup\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$.
- Note: for a regular limit, it's an equals rather than this inequality: $\lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n + b_n$.
- For $\liminf\limits_{n\to\infty} a_n + b_n \geq \liminf\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

Theorem 5.1 (Cesaro limit theorem) *If* $\lim\limits_{n\to\infty} a_n = L$ *then* $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = L$. *Opposite is not necessarily true – considering oscillating sequences that cancel each other out.*

Proof:
Take any $\epsilon > 0$, $\exists N$ such that $\forall n \geq N$, $|a_n - L| < \epsilon$. [Definition of $\lim\limits_{n\to\infty} a_n = L$]
Therefore, $\forall n \geq N$, we can rearrange terms so that $a_n < L + \epsilon$ [from $|a_n - L| < \epsilon$].
$\Rightarrow \forall n \geq N$:

$$\begin{aligned}
\frac{a_1 + ... + a_n}{n} &= \frac{a_1 + ... + a_{N-1}}{n} + \frac{a_N + ... + a_n}{n} \\
&= \text{[breaking up the sum into terms} < N \text{ and terms} \geq N\text{]} \\
&\leq \frac{a_1 + ... + a_{N-1}}{n} + \frac{(L+\epsilon) + (L+\epsilon) + ... + (L+\epsilon)}{n} \\
&\text{Since } \forall n \geq N,\ a_n < L + \epsilon \\
&= \frac{a_1 + ... + a_{N-1}}{n} + \frac{n - N + 1}{n} * (L + \epsilon) \\
&\text{There are n - N + 1 terms} \geq N \text{ and} \leq n
\end{aligned}$$

Let's say there are b_n terms in the first summation (i.e. $\frac{a_1+...+a_{N-1}}{n}$) and c_n terms in the second summation $\frac{n-N+1}{n} * (L + \epsilon)$.

Then $\limsup \frac{1}{n}\sum_{k=1}^{n} a_k \leq \limsup\,(b_n + c_n) \leq \limsup b_n + \limsup c_n$. We see $\lim\limits_{n\to\infty} b_n = 0$ since the numerator is a finite sum while the denominator $\to \infty$ and $\lim\limits_{n\to\infty} c_n = L + \epsilon$ since the term $\frac{n-N+1}{n} \to 1$ as $n \to \infty$.

Therefore, $\limsup \frac{1}{n}\sum_{k=1}^{n} a_k \leq L+\epsilon$. Similarly, $\liminf\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k \geq L$. Therefore, the limit $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k$ exists and is equal to L.

5.2 Infinite Series

Let $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers. We say that the infinite series $\sum\limits_{n=1}^{\infty} a_n$ converges if the sequence $\{\sum_{k=1}^{\infty} a_k\}_{n=1}^{\infty}$ [i.e. sum everything below n in $\{x_n\}_{n=1}^{\infty}$to represent the nth value of the infinite series] converges. Its value is declared to be the limit of the sequence $\{\sum_{k=1}^{\infty} a_k\}_{n=1}^{\infty}$.

Ex. Geometric Series.

Take some r such that $|r| < 1$. Then

$$(1-r)(1+r+r^2+...+r^n) = 1+r+r^2+...+-r(1+...+r^n) = 1-r^{n+1}$$

Note r $\neq 1$ and

$$\Rightarrow 1+r+r^2+...+r^n = \frac{1-r^{n+1}}{1-r}$$
$$= \frac{1-r^{n+1}}{1-r}$$
$$= \frac{1}{1-r} - \frac{r^{n+1}}{1-r}$$

So $\lim\limits_{n\to\infty} 1+r+..+r^n = \frac{1}{1-r}$ since r^{n+1} as $n \to \infty = 0$ as $r < 1$. Thus $\sum_{n=0}^{\infty} r_n = \frac{1}{1-r}$ if $|r| < 1$.

Theorem 5.2 *If $\sum_{n=1}^{\infty} a_n$ converges (or equivalently, exists), then $\lim\limits_{n\to\infty} a_n = 0$.*

Proof:
Let $p_n = \sum\limits_{k=1}^{n} a_k$ and $q_n = \sum_{k=1}^{n-1} a_k$. Also let $L = \sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} p_n$.

However, $q_n = p_{n-1}$ and therefore $\lim\limits_{n\to\infty} q_n = L$. We can find $a_n = p_n - q_n$, and since $\lim\limits_{n\to\infty} p_n = \lim\limits_{n\to\infty} q_n$, $\lim\limits_{n\to\infty} a_n = 0$.

Expanding on $\lim\limits_{n\to\infty} p_n = \lim\limits_{n\to\infty} q_n$: Take any $\epsilon > 0$. We can then find N such that $\forall n \geq N$, $|p_n - L| < \epsilon$. We can now define $N' = N+1$ and take any $n \geq N'$. Then we can show $\lim\limits_{n\to\infty} q_n = L = \lim\limits_{n\to\infty} p_{n-1}$.

The converse implication is **not** true: $\sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$ [i.e. does not converge] even though $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Proof: Did in class. Broke up into splits of $\sum_{2^{k-1}}^{2^k-1} a_n$ and showed each was $\leq$ than 1/2. Showed each partial sum was bounded below $\to$ the sum goes to $\frac{k}{2}$ as $k \to \infty$ therefore, the series does not converge.

Theorem 5.3 *If $a_n \geq 0\ \forall n$, and $\{\sum\limits_{k=1}^{n} a_k\}_{n=1}^{\infty}$ is bounded above [i.e. the series is bounded above], then $\sum\limits_{n=1}^{\infty} a_n$ converges. Converse is also true (only for non-negative sequences).*

Proof:
Let $b_n = \sum_{k=1}^{n} a_k$. Then b_n is an increasing sequence since $a_n \geq 0\ \forall n$ [using the definition of the series $\sum\limits_{n=1}^{\infty} a_n$ as equivalent to the sequence $b_n = \{\sum\limits_{k=1}^{n} a_k\}_{n=1}^{\infty}$].

Thus b_n converges if and only if it is a bounded sequence.

Ex.

- We'd like to prove $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges
  1. $\frac{1}{n!} \leq \frac{1}{1*2*2*2..*2} = \frac{1}{2^{n-1}}$
  2. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges since its a geometric series with $|r| = \frac{1}{2} < 1$.
  3. $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ are bounded above and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is an upper bound of $\sum_{n=1}^{\infty} \frac{1}{n!}$
  4. $\rightarrow \{\sum_{k=0}^{n} \frac{1}{k!}\}_{n=1}^{\infty}$ is bounded above.
- $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges for any $k > 1$. Riemann Zeta function.

Def.[Absolute Convergence]:
A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges. [just have to show it's bounded above because a it is a series of non-negative terms].

6 October 7

6.1 Absolute Convergence

Def.[Positive and Negative Parts]

- Take any $x \in \mathbb{R}$.
- $x^+ = \max\{x, 0\} = \begin{cases} \text{x if x} \geq 0 \\ 0\text{if } x < 0 \end{cases}$ = the posiitve component of x.
- $x^- = -\min\{x, 0\} = \begin{cases} 0 \text{ if x} > 0 \\ -x\text{if } x \leq 0 \end{cases}$ = the negative component of x.
- **Note: both x^+ and x^- are non-negative valued real numbers.**
- $x = x^+ - x^-$
- $|x| = x^+ + x^-$

Theorem 6.1 (Absolutely Convergent Series) *An absolutely convergent series is convergent.*

Proof: Let $\sum\limits_{n=1}^{\infty} a_n$ be absolutely convergent. That means $\sum\limits_{n=1}^{\infty} |a_n|$ converges. We would like to show $\sum\limits_{n=1}^{\infty} a_n$ converges.

We first consider the finite summation $\sum\limits_{k=1}^{n} a_k = \sum\limits_{k=1}^{n} (a_k^+ - a_k^-) = \sum\limits_{k=1}^{n} a_k^+$ - $\sum\limits_{k=1}^{n} a_k^-$

. Since $\sum_{n=1}^{\infty}|a_n|$ is convergent, there exists a limit L such that any finite sum is less than or equal to L: $\exists L$ such that $\forall n$, $\sum_{k=1}^{n}|a_n| \leq L$. Therefore, $\forall n$, $\sum_{k=1}^{n} a_k^+ + \sum_{k=1}^{n} a_k^- = \sum_{k=1}^{n}|a_k| \leq L$ and both summations $\sum_{k=1}^{n} a_k^+ \leq L$ and $\sum_{k=1}^{n} a_k^- \leq L$ are individually bounded above by L $\Rightarrow$ both converge as $n \to \infty$.
This last part uses the theorem from last class:
If $a_n \geq 0 \forall n$, and $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ is bounded above, then $\sum_{n=1}^{\infty} a_n$ converges. Converse is also true (only for non-negative sequences).

Therefore $\lim_{n\to\infty}\sum_{k=1}^{n} a_k^+$ and $\lim_{n\to\infty}\sum_{k=1}^{n} a_k^-$ both exist [although we don't know what they are].

Ex. $\sum_{n=1}^{\infty}\frac{1}{n!} \Rightarrow \sum_{n=1}^{\infty}|\frac{1}{n!}| = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}$ and this converges $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n!}$ converges!

6.2 Conditional Convergence and Alternating Series

Def.[Conditionally Convergent]. A series is called conditionally convergent if it is convergent, but not absolutely convergent.
Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ...$ converges to $\log(2)$ but is not absolutely convergent.

Theorem 6.2 (Alternating Series) *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence converging to 0. Then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + ...$ converges (i.e. the alternating series converges).*

Proof:[sketch]: $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 =$

$$\begin{cases} (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) \text{ where } (a_1 - a_2) \geq 0,\ (a_3 - a_4) \geq 0,\ ... \\ a_1 - (a_2 - a_3) - (a_4 - a_5) - a_6 \leq a_1 - a_6 \text{ where } (a_2 - a_3) \geq 0 \text{ and } (a_4 - a_5) \geq 0 \end{cases}.$$

Let $S_n = \sum_{k=1}^{2} n(-1)^{k-1}a_k \Rightarrow S_n = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + ... + (a_{2n-1} - a_{2n})$. Then $\{S_n\}_{n=1}^{\infty}$ is an increasing sequence [and is composed of the even terms of the original series]. On the other hand, $S_n = (a_1 - (a_2 - a_3) - (a_4 - a_5) - ... - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1 - a_{2n} \leq a_1$. Thus, we have found a bound for S_n which is an increasing sequence. Therefore, we have a limit for S_n: $\lim +n \to \infty S_n$ exists, and we'll call it L.

Define $t_n = \sum_{k=1}^{n}(-1)^k a_k$. We need to show $\lim n \to \infty t_n$ exists. **Claim:** $\lim n \to \infty t_n = L$.
Proof Take any $\epsilon > 0$.

- $\exists N_1$ such that $\forall n \geq N_1$, $|S_n - L| < \frac{\epsilon}{2}$.
- $\exists N_2$ such that $\forall n \geq N_n$, $|a_n| < \frac{\epsilon}{2}$. [This arrives from $a_n \to 0$ as $n \to \infty$].

Let $N = \max\{2N_1 + 1, N_2\}$, take any $n \geq N$. Then there are two cases:

- n is even. Then $t_n = S_{n/2}$. Note that $\frac{n}{2} \geq \frac{N}{2} \geq N_1 \Rightarrow |t_n - L| = |S_{n/2} - L| < \frac{\epsilon}{2}$.
- n is odd. Then n = 2m + 1 where $m = \frac{n-1}{2}$. Therefore, $t_n = s_m + a_n$ and
  - $n \geq N \Rightarrow m \geq N_1 \Rightarrow |S_m - L| < \frac{\epsilon}{2}$
  - and $n \geq N \Rightarrow n \geq N_2 \Rightarrow |a_n| < \frac{\epsilon}{2}$

  Thus, $|t_n - L| < \epsilon$. [from $t_n = a_1 - a_2 + ... + a_{n-2} - a_{n-1} + a_n$ where $a_1 - a_2 + ... + a_{n-2} - a_{n-1} = S_{(n-1)/2}$ so $|S_m + a_n - L| \leq |S_m - L| + |a_n| < \epsilon$].

Theorem 6.3 *Let $\sum\limits_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then for any bijection $\pi : N \to N$, $\sum\limits_{n=1}^{\infty} a_{\pi(n)}$ is also absolutely convergent and the two values are equal: $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a_{\pi(n)}$. This effectively says reordering the terms of the series doesn't change what it converges to.*

6.3 Metric Spaces

Def.[Metric Spaces] Let M be a set. A metric on M is a function d: from $d : MxM \to [0, \infty)$ [think of this like a distance function] which satisfies

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)\ \forall x, y \in M$.
3. $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in M$

Then the pair (M,d) is called a metric space.

Ex.

- Let $M = \mathbb{R}$, $d(x, y) = |x - y|$
- $M = \mathbb{R}^k = \{(x_1, ..., x_k)|x_1, ..., x_k \in \mathbb{R}\}$. $d(x, y) = \sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}$ [Euclidean Distance].
- $M = \mathbb{R}^k$. $d(x, y) = \sum_{i=1}^{k} |x_i - y_i|$ (l^1 metric – I know this as L_1).
- $M = l^2 = \{(x_1, x_2, ...)|x_i \in \mathbb{R}, \forall i, \sum_{i=1}^{\infty} x_i^2$ converges$\}$. $d(x, y) = \sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}$.
- $M = l^1 = \{(x_1, x_2, ...)|x_i \in \mathbb{R}, \forall i, \sum_{i=1}^{\infty} |x_i|$ converges$\}$. $d(x, y) = \sum_{i=1}^{k} |x_i - y_i|$.

7 October 10

7.1 Euclidean Distance is a metric on $\mathbb{R}$

We left off with needing to prove the triangle inequality to prove that Euclidean distance $d(x,y) = \sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}$ is indeed a metric on $M = \mathbb{R}^n$. We begin with proving the Cauchy-Schwarz Inequality.

Theorem 7.1 (Cauchy-Schwarz Inequality) *Let* $a_1, ..., a_n,\ b_1, ..., b_n \in \mathbb{R}$. *Then* $|\sum\limits_{i=1}^{n} a_i b_i| \leq \sqrt{\sum\limits_{i=1}^{n} a_i^2 \sum\limits_{i=1}^{n} b_i^2}$

Proof:

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 \text{ [square of difference is non-negative]} \\ &= \sum_{i}\sum_{j}(a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i b_j a_j b_i) \text{ [expanding out terms...]} \end{aligned}$$

Note that

$$\begin{aligned} \sum_{i}\sum_{j} a_i^2 b_j^2 &= (\sum_{i} a_i^2)(\sum_{j} b_j^2) \\ &= (\sum a_i^2)(\sum b_i^2) \\ &\quad \text{since summation starts at } i = j = 1 \text{ and goes to n} \end{aligned}$$

And similarly,

$$\sum_{i}\sum_{j} a_j^2 b_i^2 = (\sum a_i^2)(\sum b_i^2)$$

Therefore, ...

$$\begin{aligned} \sum_{i}\sum_{j} a_i b_j a_j b_i &= (\sum_{i} a_i b_i)(\sum_{j} a_j b_j) \\ &= (\sum_{i} a_i b_i)(\sum_{i} a_i b_i) \\ &= (\sum a_i b_i)^2 \end{aligned}$$

Finally, we see that

$$\begin{aligned} 0 &\leq \sum_{i}\sum_{j}(a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i b_j a_j b_i) \\ \Rightarrow 0 &\leq 2(\sum a_i^2)(\sum b_i^2) - 2(\sum a_i b_i)^2) \\ \Rightarrow (\sum a_i b_i)^2 &\leq (\sum a_i^2)(\sum b_i^2) \end{aligned}$$

We can now proceed with the triangle inequality proof:

$$\begin{aligned}
\sum_{i=1}^{n}(a_i+b_i)^2 &= \sum_i a_i^2 + \sum_i b_i^2 + 2\sum_{i=1}^{n}(a_i+b_i) \\
&\leq \sum_{i=1}^{n}a_i^2 + \sum_{i=1}^{n}b_i^2 + 2\sqrt{\sum_{i=1}^{n}a_i^2 + \sum_{i=1}^{n}b_i^2} \\
&\text{Note: } \sum_{i=1}^{n}(a_i+b_i) \leq \sqrt{\sum_{i=1}^{n}(a_i+b_i)^2} \leq \sqrt{\sum_{i=1}^{n}a_i^2 + \sum_{i=1}^{n}b_i^2} \\
&= (\sqrt{\sum a_i^2} + \sqrt{\sum b_i^2}) \\
&\Rightarrow \sqrt{\sum (a_i+b_i)^2} \leq \sqrt{\sum a_i^2} + \sqrt{\sum b_i^2}
\end{aligned}$$

Formally [proving the triangle inequality for euclidean space]:
Take $x, y, z \in \mathbb{R}^n$, $x = (x_1, ..., x_n)$, etc. Then

$$\begin{aligned}
d(x,z) &= \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \\
&= \sqrt{\sum_{i=1}^{n}(x_i - y_i + y_i - z_i)^2} \\
&\leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(z_i - y_i)^2} \\
&= d(x,y) + d(y,z)
\end{aligned}$$

7.2 l^2 Metric Space

Recall we define a set $M = l^2 = \{(x_1, x_2, ...) : x_i \in \mathbb{R}, \forall i, \sum_{i=1}^{\infty} x_i^2 < \infty\}$. If $x, y \in l^2$, $d(x,y) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$, and (M,d) is the metric space.

Fact. $d(x,y) < \infty \ \forall x, y \in l^2$.
Proof:
For any n, $\sum\limits_{i=1}^{n}(x_i - y_i)^2 \leq \left[\sqrt{\sum\limits_{i=1}^{n}x_i^2} + \sqrt{\sum\limits_{i=1}^{n}y_i^2}\right]^2 \leq \left[\sqrt{\sum_{i=1}^{\infty}x_i^2} + \sqrt{\sum_{i=1}^{\infty}y_i^2}\right]^2 < \infty$ since x and y are in l^2.

Triangle inequality in l^2 is the same: Let $x, y \in l^2$. Then

$$\begin{aligned}
d(x,y) &= \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2} \\
&= \sqrt{\left[\lim_{n\to\infty}\sum_{i=1}^{n}(x_i - y_i)^2\right]}
\end{aligned}$$

Since we have a finite sum, we can break it up into two finite parts...

$$\begin{aligned}
&= \sqrt{\lim_{n\to\infty}\left[\sum_{i=1}^{n}(x_i - z_i + z_i - y_i)^2\right]} \\
&\leq \sqrt{\lim_{n\to\infty}\left[\sum_{i=1}^{n}(x_i - z_i)^2\right]} + \sqrt{\lim_{n\to\infty}\left[\sum_{i=1}^{n}(z_i - y_i)^2\right]} \\
&= \lim_{n\to\infty}\sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} + \lim_{n\to\infty}\sqrt{\sum_{i=1}^{n}(z_i - y_i)^2} \\
&= d(x,z) + d(z,y)
\end{aligned}$$

In this logic, we used

- $a_n \to a \Rightarrow \sqrt{a_n} \to \sqrt{a}$
- $a_n^2 \to a_2$, $a_n \leq b_n \forall n \Rightarrow \lim a_n \leq \lim b_n$

7.3 Sequences in metric spaces

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in a metric space (M,d).

We say that $\lim\limits_{n\to\infty} x_n = x$ (i.e. $x_n \to x$), if $\forall \epsilon > 0$, $\exists N$ such that $\forall n \geq N$, $d(x, x_n) < \epsilon$.

Notably, we simply use distance to the limit rather than the difference of the value of the reals (i.e. order axiom) for the definition of convergence.

Fact. If the limit exists, then it is unique.
Note $d(x,y) = 0 \iff x = y$ from our definition of a distance metric on a set satisfying a metric space. Therefore, if we have two limits L_1, L_2 acting on a sequence, then the distance $d(L_1, L_2)$ must be 0. From that, we can conclude the distance must be the same.

7.4 Convergence in $\mathbb{R}^n$

Let $\{x^{(k)}\}_{k=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$. Then $\lim_{k\to\infty} x^{(k)} = x$ if and only if for each $i = 1, ..., n$ $\lim_{k\to\infty} x_i^{(k)} = x_i$. More simply, each ith coordinate in $x_i^{(k)}$ converges to a value x_i.

Proof:
Suppose that $\lim_{k\to\infty} x^{(k)} = x$. Take any $i \in \{1, ..., n\}$ and take any $\epsilon > 0$, Then $\exists N$ such that $\forall k \geq N$, $d(x^{(k)}, x) < \epsilon$.

We take any $k \geq N$, then

$$
\begin{aligned}
|x_i^{(k)} - x| &= \sqrt{(x_i^{(k)} - x_i)^2} \\
&\leq \sqrt{\sum_{i=1}^{n} (x_j^{(k)} - x_j)^2} \\
&= d(x^{(k)}, x) \\
&< \epsilon
\end{aligned}
$$

Thus $x_i^{(k)} \to x_i$ as $k \to \infty$. If the sequence converges, then the coordinates must converge.

Conversely, suppose that for each $i = 1, ..., n$, the $\lim_{k\to\infty} x_i^{(k)} = x_i$ (i.e. each of the coordinates converge).

Take any $\epsilon > 0$, for each i, $\exists N_i$ such that $\forall k > N_i$, $|x_i^{(k)} - x_i| < \frac{\epsilon}{\sqrt{n}}$. [Each coordinate converges to x_i].

Then let N = $\max\{N_1, N_2, ...N_n\}$. Take any $k \geq N$, then

$$
\begin{aligned}
d(x^{(k)}, x) = &\sqrt{\sum_{i=1}^{n} (x_i^{(k)} - x_i)^2} \\
&\text{Euclidean distance measure} \\
&< \sqrt{\sum_{i=1}^{n} \frac{\epsilon^2}{n}} \\
&\text{Each coordinate is bounded by } \frac{\epsilon}{n} \\
&= \epsilon
\end{aligned}
$$

Very simple: if each coordinate is bounded by x_i, then the series is bounded by x = $[x_1, x_2, ..., x_n]$.

Ex. Consider the set l^2. Let $x^{(k)} = (0, 0, 0, ..., 0, 1, 0, 0, ...)$ where there is a 1 at the k^{th} coordinate and 0 everywhere else. Clearly, $x \in l^2$.

So take any i, $\lim_{k\to\infty} x_i^{(k)} = 0$. On the other hand $x^{(k)} \not\to (0, 0, ..., 0) = 0$. So $d(x^{(k)}, \vec{0}) = 1$. Clearly, $x^{(k)}$ is not converging to 0, and we'll see later the sequence is not converging to any point in l^2. This is one reason l^2 is fundamentally different to $\mathbb{R}$: it is not "complete" (why we call Axiom 13 of $\mathbb{R}$ the "completeness axiom"). If we have a sequence of values in a set that converges outside this set, the set is not "complete".

Metric spaces allow us to define the framework to study function spaces [i.e. spaces where every point in the space is a function]. The notion of "distance" is then conveyed as a distance between functions. In a sense l^2 is a function space: you can apply a fourier expansion to a function, and the infinite fourier coefficients will reside in l^2. We can therefore study the fourier series as an infinite sequence in l^2.

Theorem 7.2 *Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in a metric space that converges to some point. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence meaning that $\forall \epsilon > 0$, $\exists N$ such that $\forall m, n \geq N$, $d(x_n, x_m) < \epsilon$.*

Note The converse is not always true!
Simple example: take metric space M to be the interval M = (0,1) with d(x,y) = $|x, y|$. The sequence $x_n = \frac{1}{n}$ is Cauchy in $\mathbb{R}$, but does not converge in M.

Rather it converges to a value outside M which is a priori not defined. This has an important consequence: Cauchy sequences in metric spaces may not have limits, and this is why we call Axiom 13 of $\mathbb{R}$the completeness axiom. Notably, it "completes" the space of $\mathbb{R}$by ensuring every Cauchy Sequence converges to a value in $\mathbb{R}$. However, as the theorem states, if you have a metric space, and a sequence in that space that is convergent to some point, then that sequence is Cauchy.

7.5 Closed Sets

Let (M,d) be a metric space and let $X \subseteq M$ be a subset. We say that a point $x \in M$ is a limit point of X if $\exists$ a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$.

Effectively this definition defines a subset of a set M and looks at the sequences of values of X that converge to a point in X.

Ex.
$M = \mathbb{R}$, X = (0,1) then 0 and 1 are limit points of X.

Def.[Limit Point] We denote the set of all limit points of X by $\bar{X}$ and denote $\bar{X}$ as the "closure" of X.

Fact
$X \subseteq \bar{X}$ since for any $x \in X$, the sequence $x, x, x, x, ...$ converges to x. Therefore every point in the set itself is an element of the closure [there's always a constant sequence composed entirely of $x \in X$ that converges to x].

Def. We say that a set X is closed if $X = \bar{X}$. All limit points are in the set itself. This may not always be the case: notably when a sequence $\{x_n\}_{n=1}^{\infty}$converges to a point $\notin X$.

Ex. Any $[a, b] \subseteq \mathbb{R}$ is closed.
Proof: Take any sequence $\{x_n\}_{n=1}^{\infty}$ in $[a, b]$. Then $a \leq x_n \leq b\ \forall n$.
$\Rightarrow a \leq x \leq b$ so $x = \lim_{n\to\infty} x_n$.

Ex. The open interval $(a, b) \subseteq \mathbb{R}$ is not closed as we can construct a sequence entirely in X that converges to the boundaries of the interval.

8 October 12

8.1 Properties Closed Sets

Def Closed Set. Let (M,d) be a metric space. A set $X \subseteq M$ is called closed if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X converging to some $x \in M$, then $x \in X$.
Ex. $[a, b]$ is a closed subset of $\mathbb{R}$ under the usual metric (absolute value distance).

- In any metric space M, the sets $\{\}$ and M are closed. Empty set is vacuously true [it contains no sequences]. If A implies B, and A is never true, than the previous statement is always true [vacuously true].
- For any $x \in M$, the set $\{x\}$ is closed. [converges to x trivially].
- The closure of any set is closed [the closure of the closure is the closure itself].

  Proof:
  Take any $X \subseteq M$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in the closure $\bar{X}$ converging to $x \in M$. Have to show this little $x \in M$ is also in the closure. Since $x_n \in \bar{X}$, there is a sequence of points in X that converges to x_n. So we can find $y_n \in X$ such that $d(x_n, y_n) = \frac{1}{n}$. Can easily prove from this that $y_n \to x$ (y_n also converges to x). Thus $x \in \bar{X}$.

Theorem 8.1 *If A and B are closed, so is $A \cup B$.*

Proof: Take a sequence $\{x_n\}_{n=1}^{\infty}$in $A \cup B$ converging to $x \in M$. Then at least one of the following must be true:

1. either $x_n \in A$ for infinitely many n
2. or $x_n \in B$ for infinitely may n.
3. (either in A or B or both – both may be true, but at least one of them is true).

If (1) is true, then we can find a $n_1 < n_2 < ...$ such that $x_{n_k} \in A \ \forall k$ [use subsequence so that we pick elements of the sequence just in A]. But $\{x_{n_k}\}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$and $x_n \to x$, so $x_{n_k} \to x$ as $k \to \infty$. And $x \in A$ because A is closed. Similarly if (2) is true, then $x \in B$. Therefore, since A and B are both closed $A \cup B$ is closed.

Corollary 8.1.1 *Any union of finitely many closed sets is closed.*

Corollary 8.1.2 *Any finite subset of a metric space is closed.*

Theorem 8.2 *Let $\mathcal{F}$ be any collection of closed sets. Then $\cap_{F \in \mathcal{F}} F$ is closed. Arbitrary intersections of closed sets is closed.*

Proof:
Take any sequence in this set, then any sequence in the intersection will be an element in all of the closed sets. Therefore, it will converge to an element in the closed set.

Ex. The Cantor set is closed. Cantor set $[0, 1/3] \cup [2/3, 1] \Rightarrow [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$. Intersection of all of these sets is the Cantor sets. Take sequence of numbers in Cantor Set, then can use the intersection of closed sets theorem.

8.2 Open Sets

Def.[Open Sets.] Given any $x \in M$ and $\epsilon > 0$, let $\beta_\epsilon(x) = \{y \in M | d(x, y) < \epsilon\}$.

$\beta_\epsilon(x)$ is called the "open ball " of radius ϵ centered at x. Also called the open ϵ-neighborhood of x.

A set $U \subseteq M$ is called "open" if for any $x \in U$, $\exists \epsilon > 0$ such that $\beta_\epsilon(x) \subseteq U$. Take any point in the set, small enough epsilon such that the open ball of radius epsilon is contained within U.

Fact. $\{\}$ and M are open. Both of these are open and closed. In $\mathbb{R}$, these are the only two sets that are both "open" and "closed".

Proposition 8.2.1 *Any open ball is an open set.*

Proof:
Take any open ball $\beta_\epsilon(x)$. Take any $y \in \beta_\epsilon(x)$. Let r = d(x,y). So then $r < \epsilon$. Let $\delta = \epsilon - r > 0$.
Claim $\beta_\delta(y) \subseteq \beta_\epsilon(x)$.
Proof: Take any $z \in \beta_\delta(y)$, then $d(x,z) \leq d(x,y) + d(y,z) < r + \delta$. $r + \delta = \epsilon$, so $d(x,z) < \epsilon$.

Corollary 8.2.1 *Any open interval $(a,b) \subseteq \mathbb{R}$ is open.*

Proof $(a,b) = \beta_{\frac{a-b}{2}}(\frac{a+b}{2})$: open ball and therefore is open. Have to show there's a smaller open interval contained within a bigger open interval.

Theorem 8.3 *A set $X \subseteq M$ is open if and only if X^C is closed. $X^C = M \backslash X = \{x \in M | x \notin X\}$. Denoted by X' in the text.*

Proof: Suppose that X is open. Take any sequence $\{x_n\}_{n=1}^{\infty}$in X^C converging to $x \in M$. Suppose that $x \notin X^C$. Then $x \in X$. Since X is open, $\exists \epsilon > 0$ such that $\beta_\epsilon(x) \subseteq X$. But for large enough n, $d(x, x_n) < \epsilon$ [definition of convergence, therefore closed set]. i.e. $x_n \in \beta_\epsilon(x) \subseteq X$. Therefore, we have a contradiction.

Conversely, suppose X^C is closed. We'd like to prove that X is not open. Then $\exists x \in X$ such that $\forall \epsilon > 0$, $\beta_\epsilon(x) \not\subseteq X$. Then $\beta_1(x) \notin X$. Choose $x_1 \in \beta_1(x) \cap X^C$. Similarly, $\beta_{\frac{1}{2}}(x) \not\subseteq X$. Choose $x_2 \in \beta_{\frac{1}{2}}(x) \cap X^C$. Can continue: $x_n \in \beta_{\frac{1}{n}}(x) \cap X^C$. Therefore $\{x_n\}$ is a sequence in X^C. And $d(x, x_n) < \frac{1}{n}$. So $x_n \to x$. Since X^C is closed, $x \in X^C \to$ Contradiction. Limit of x_n is in X^C since it's a closed set.

9 October 14

9.1 Properties of Open Sets

Finite union of closed sets is closed, intersection of closed sets is closed. What's the appropriate statement for open sets?

1. Any finite intersection of open sets is open.
2. Any arbitrary union of open sets is open.

Facts
(1) If $U_1, U_2, ...U_n$ are open [i.e. finite number of intersection of open sets], Then so is $U_1 \cap ... \cap U_n$.
Proof: $(U_1 \cap ... \cap U_n)^C = U_1^C \cup ... \cup U_n^C$. This is closed since $U_1^C, .., U_n^C$ is closed.

(2) If $(U_\alpha)_{\alpha \in \mathcal{A}}$ is any collection of open sets, then so is $\cup_{\alpha \in \mathcal{A}} U_\alpha$.
Proof $(\cup_{\alpha \in \mathcal{A}} U_\alpha)^C = \cup_{\alpha \in A} U_\alpha^C$ is closed since each U_α^C is closed.

9.2 Compact Sets

Compact Sets: Notion of compactness is one of the great advances of modern math.

Def.[Compact Set]:Let (M,d) be a metric space. Let $X \subseteq M$ be a subset of M. Let $(U_\alpha)_{\alpha\in\mathcal{A}}$ be a collection of open subsets of M. We say that this is an **open cover** of X if $X \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$.

Let $M = \mathbb{R}$ and $X = \{1, 2\}$, then

1. $\{(-0, 3)\}$ is an open cover of X.
2. $\{(1/2, 3/2), (3/2, 5/2)\}$ is also an open cover.
3. $\{(0, 2), (1, 5)\}$ [the intervals in a set can overlap – no need to be disjoint].

Def. A set $X \subseteq$ M is called "compact" if every open cover $\mathcal{U}$ of X has a finite subcover (i.e. a finite set $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is also an open cover of X). [V is a finite subcollection which covers X. U may be infinite]. Have to consider all open covers of X and that there exists a finite subcover...

Theorem 9.1 *A set $X \subseteq M$ is compact $\iff$ (if and only if) for any sequence $\{x_n\}_{n=1}^\infty \in X$, there is a subsequence converging to a point in X.*

We proved this for closed intervals on the Real Line. But this generalizes that notion to arbitrary metric spaces. Same criterion of compactness.

Compact is the generalization of "boundedness" to metric spaces that are not the real line.

Proof:
First suppose that X is compact. Take any sequence $\{x_n\}_{n=1}^\infty \in X$. We have to show it has a convergent subsequence to a point in X.

Suppose that $\{x_n\}_{n=1}^\infty$ has no subsequence that converges to a point in X.
Claim: For all $x \in X$, $\exists \epsilon_x > 0$ such that $\beta_{\epsilon_x}(x)$ [small enough open ball around x which...] contains x_n for only finitely many n.
Proof: Suppose not. Then $\exists x \in X$ such that $\forall \epsilon_x > 0$, β_{ϵ_x} contains x_n for infinitely many n [negating the above claim]. There exists a point (i.e. x) such that there is a ball around it which contains x_n for infinitely many n. So $\exists n_1$ such that $x_{n_1} \in \beta_1(x)$, $\exists n_1 > n_2$ such that $x_{n_2} \in \beta_{1/2}(x)$, $\exists n_3 > n_2$ such that $x_{n_3} \in \beta_{1/3}(x)$ and so on...
So $\{x_n\}_{n=1}^\infty$ is a subsequence such that $d(x, x_{n_k}) < \frac{1}{k} \Rightarrow x_{n_k} \to x$ Therefore contradiction [$\nexists \beta_\epsilon(x)$ that contains x_n for infinitely many n since $x_{n_k} \to x$].

Now returning to the original proof now that this claim has been proved...
Recall that any open ball is an open set [proved in the last lecture]. So the collection $\{\beta_{\epsilon_x}(x)\}_{x\in X}$ is an open cover of X. Any little x belongs in β_{ϵ_x}. Since X is compact it has a finite subcover $B_{\epsilon_{y_1}}(y_1), ..., \beta_{\epsilon_{y_1}}(y_k)$ so that the balls around them is a cover of X. Each of these balls contains x_n for only finitely many n. And their union contains X. This is impossible, since X contains x_n $\forall n$. Finitely many balls such that their union is an infinite set. Therefore, if X is a compact set, we've shown there's a convergent subsequence in x that converges within X. [It's not the point – rather the indices which are causing the contradiction: finitely many x_n in the subsets: i.e. x_1, x_5, x_{21} is in one of these balls $\beta_{\epsilon_{y_1}}(y_1)$. Finitely many balls, so one of them must contain x_n for some n which goes to infinity $\rightarrow$ contradiction].

($\Rightarrow$) Conversely, suppose that every sequence in X has a subsequence that converges to a point in X.

To show: We want to show X is compact.
Let U be any open cover of X.
To show: U has a finite subcover. There's always an open cover of any set [take the full space]. Any set that has a finite open cover is vacuously compact.

There are two steps in the proof:
Step 1: $\exists\epsilon > 0$ such that $\forall x \in X$, $\exists U \in \mathcal{U}$ such that $\beta_\epsilon(x) \subseteq U$. U is an open cover, so any x is contained in one element of U, and there's a ball around x containing that x. Claim is even stronger, we have a single ϵ that works for all x (not simply each x has its own ϵ_x for the radius of the open ball). $\mathcal{U}$ is the collection of open sets whose union contains the entire set X. Some little ball around x which contains x. and our claim here is that there's some ϵ that works for all x. And that's where we use the assumption that every sequence has a convergent subsequence.

10 October 17

Theorem 10.1 (Bolzano-Weierstrauss Theorem for Metric Spaces) *A subset $X \subseteq M$ is compact if and only if every sequence in X has a subsequence that converges to a point in X.*

($\Rightarrow$) We proved that if X is compact, then every sequence has a convergent subsequence.
($\Leftarrow$) Remains to prove the converse [if every point has a subsequence that converges to within X, then X is compact].
Suppose that every sequence in X has a subsequence that converges to a point in X. Let $\mathcal{U}$ be an open cover for X.
Step 1: $\exists\epsilon > 0$ such that $\forall x \in X$, $\exists u \in \mathcal{U}$ with $\beta_\epsilon(x) \subseteq \mathcal{U}$. [$\epsilon$ does not depend on x! Open set around x of size ϵ that's a subset of our open cover.].

Suppose not. Then $\forall \epsilon > 0$, $\exists x_\epsilon \in X$ such that $\forall U \in \mathcal{U}$, $\beta_\epsilon(x_\epsilon) \not\subseteq U$.
Let $y_n = x_{\frac{1}{n}}$. Then $y_n \in X\ \forall n$, $\beta_{\frac{1}{n}}(y_n) \not\subset U$ for any $U \in \mathcal{U}$. Since $\{y_n\}_{n=1}^{\infty}$ is a sequence in X, there is a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ converging to some $y \in X$ [use hypothesis that every sequence is a convergent subsequence].
Since $y \in X$, $\exists U \in \mathcal{U}$ [Union of all Us contain X] such that $y \in U$ [union of all the Us contain X, so y belongs to some U]. Since U is open, $\exists \epsilon > 0$ such that $\beta_\epsilon(y) \subseteq U$ [any ball around y is fully contained in the open set]. Choose k so large that $d(y, y_{n_k}) < \frac{\epsilon}{2}$ and $\frac{1}{n_k} < \frac{\epsilon}{2}$.

Take any $z \in \beta_{\frac{1}{n_k}}(y_{n_k})$ Then $d(y,z) \leq d(y, y_{n_k} + d(y_{n_k}, z) < \epsilon/2 + 1/n_k < \epsilon$ $\Rightarrow z \in \beta_\epsilon(y)$. Thus, $\beta_{\frac{1}{n_k}}(y_{n_k}) \subseteq U$. Thus, we have a contradiction.

Step 2: Given any $\epsilon > 0$, $\exists$ a finite collection of points $x_1, ..., x_n \in X$ such that $X \subseteq \cup_{i=1}^{n} \beta_\epsilon(x_i)$.
Proof:
Suppose not. Then $\exists \epsilon > 0$ such that $X \notin \cup_{i=1}^{n} \beta_\epsilon(x_i)$ [X is not contained in the union of epsilon-raidus balls around x_i for a finite subset of x_i in X. Take any $x_1 \in X$ [if X is empty, then everything is true vacuously]. Since $X \not\subseteq B_\epsilon(x_1)$, $\exists x_2 \in X$ such that $x_2 \notin \beta_\epsilon(x_1)$. That is $d(x_1, x_2) \geq \epsilon$. But $X \notin \beta_\epsilon(x_1) \cup \beta_\epsilon(x_2)$ so $\exists x_3 \in X$ such that $x_3 \notin \beta_\epsilon(x_1) \cup \beta_\epsilon(x_2)$
$\Rightarrow d(x_1, x_3) \geq \epsilon$, $d(x_2, x_3) \geq \epsilon)$
Proceeding inductively, we can get an infinite sequence $x_1, x_2, ... \in X$ such that $d(x_i, x_j) \geq \epsilon\ \forall i \neq j$.This implies that no subsequence of $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Completing the proof
Let $\mathcal{U}$ be an open cover of X. By Step 1, $\exists \epsilon > 0$ such that $\forall x \in X$, $\exists U \in \mathcal{U}$ with $\beta_\epsilon(x) \subseteq U$. By Step 2, $\exists x_1, ..., x_n \in X$ such that $X \subseteq \cup_{i=1}^{n} \beta_\epsilon(x_i)$ where ϵ is from Step 1. So by Step 1, $\exists U_1, U_2, ...U_n \in \mathcal{U}$ such that $\beta_\epsilon(x_1) \subseteq U_1, \beta_\epsilon(x_2) \subseteq U_2, ..., \beta_\epsilon(x_n) \subseteq U_n$. $\Rightarrow X \subseteq \cup_{i=1}^{n} \beta_\epsilon(x_i) \subseteq \cup_{i=1}^{n} U_i$. Therefore, X is a compact set.

Theorem 10.2 (Bolzano-Weierstrauss Therorem for $\mathbb{R}^n$) *Any bounded sequence [contained in a large ball] of points in $\mathbb{R}^n$ has a convergent subsequence. A bounded sequence means $\exists \mathbb{R}$ such that $||x^{(k)}|| \leq \mathbb{R}\ \forall k$.*

Proof for $\mathbb{R}^2$:
Suppose we have $(x_1, y_1), (x_2, y_2), ... \in \mathbb{R}^2$ is a bounded sequence. Then $x_1, x_2, ...$ is a bounded sequence in $\mathbb{R}$, so it has a convergent subsequence $x_{n_1}, x_{n_2}, ...$ where $n_1 < n_2 < ...$ is an increasing sequence of positive integers. But $y_1, y_2, ...$ is also a bounded sequence in $\mathbb{R}$. Thus $y_{n_1}, y_{n_2}, ...$ is a bounded sequence in $\mathbb{R}$ [using the same $n_1, n_2, ...$ from the X]. Thus, there is s sub-sub-sequence $y_{n_{k_1}}, y_{n_{k_2}}, ...$ which converges. But $x_{n_{k_1}}, x_{n_{k_2}}$ also converges because it is a subsequence of a convergent sequence [and subsequences of convergent sequences converge]. So we've proved we have convergence in $\mathbb{R}^n \iff$ we have coordinate-

wise convergence. So $(x_{n_{k_1}}, y_{n_{k_1}}), (x_{n_{k_2}}, y_{n_{k_2}})$ converges in $\mathbb{R}^2$. And so on for $\mathbb{R}^n$...

If you have an infinite sequence of infinite sequences, can find a subsequence of which each of the infinite sequences converges [diagonalization Cantor argument].

Theorem 10.3 (Heine-Borel Theorem for $\mathbb{R}^n$) *A subset $X \in \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof:
Suppose X is closed and bounded. Take any seq $\{x_n\}_{n=1}^{\infty}$in X. By Bolsano-Weie, it has a convergence subsequence. Since X is closed the limit of this subsequence $\in X$. This shows that any sequence in has a convergent subsequence in X. This implies that X is compact [from the generalized Bolsano-Weie Thm we just proved].

10.1 October 19

Theorem 10.4 (Heine-Borel Theorem for $\mathbb{R}^n$) *A subset $X \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

We proved that if $X \subseteq \mathbb{R}^n$ is closed and bounded, then it is compact (using Bolzano-Weierstrauss Theorem for $\mathbb{R}^n$ and for metric spaces). **This is generally not true for infinite metric spaces: only $\mathbb{R}^n$.**

For the opposite direction, it suffices to use the following general result.

Theorem 10.5 *Any Compact subset of a metric space is closed and bounded.*

Bounded means that $\exists R \geq 0$ such that $d(x, y) \leq R \ \forall x, y \in M$ [M is the metric space].
Proof:
Let $X \subseteq M$ be compact. Take any sequence $\{x_n\}_{n=1}^{\infty}$in X converging to some point $x \in M$. We need to show the limit of $\{x_n\}_{n=1}^{\infty}$is in X: need to use the fact that X is compact to get that. **[Practice for the midterm: think about what theorems you can use to prove this.]**. By Bolzano-Weierstrauss theorem for metric spaces, $\exists$ a subsequence $\{x_{n_k}\}$ that converges to some point $y \in X$ [result of compactness]. Next step: show that $x = y$.
Easy Fact: If a sequence in a metric space converges, then any subsequence converges to the same limit. Follows immediately from the definition of limit.
Thus, $y = x$ and $x \in X$. So any compact set is closed necessarily.

Now we need to prove that it's bounded. Take any $x \in X$ [if empty, then vacuously true], then the set $\{\beta_n(x) : n = 1, 2, ...\}$ covers X (set of open balls centered at X) [fact of compact sets]. So this has a finite subcover: $\beta_{n_1}(x), ..., \beta_{n_k}(x)$ [why

is this collection of open balls a cover for X?] Formal way to prove it is to take any $y \in X$, we need to show y is in one of these balls: $d(x, y) < k$, then can find a finite subcover:
Let n = $\max\{n_1, ..., n_k\}$, then these balls are becoming larger and then $X \subseteq \beta_n(x)$ [there is exactly one ball that covers X]. This implies $d(x, y) < n \; \forall y \in X$ $\Rightarrow d(y, z) \leq d(y, x) + d(x, z) < 2n$ [using the definiton of open set for distances].

Theorem 10.6 *Any closed subset of a compact set is compact.*

Proof
Let $X \subseteq M$ be compact and $F \subseteq X$ be closed. Take any sequence $\{x_n\}_{n=1}^{\infty}$ in F. We have to prove there's a subsequence that converges to a point in F: $\exists$ subsequence $\{x_{n_k}\}$ converges to a limit $x \in F$.

Since $F \subseteq X$, $\{x_n\}_{n=1}^{\infty}$is also a sequence in X. Thus, by the BW thm, $\exists$ a subsequence $\{x_{n_k}\}$ that converges to some $x \in X$. Last step: $x \in F$. But this subsequence $\{x_{n_k}\}$ is in F, and F is closed, so $x \in F$. Therefore, F is also compact.

Ex.
Cantor set is compact.

Def.[Relative Metric]
Let (M, d) be a metric space and X be a subset of M. Then (X, d) is also a metric space, and d is called the "relative metric" on X.
Note: X is always an open subset in the metric space (X,d), but may not be open in (M,d). X is always open and always closed in (X,d), but may not necessarily be closed or open in the context of (M,d).

Theorem 10.7 *A set $A \subseteq X$ is open in (X,d) if and only if $A = X \cap U$ for some open set U in (M,d)*

Proof
Suppose $A \subseteq X$ is open in (X,d). We need to produce the set U.
Take any $a \in A$, then $\exists \epsilon_a > 0$ such that $B^X_{\epsilon_a}(a) = \{y \in X | d(a, y) < \epsilon_a\} \subseteq A$. Open ball around a in metric space X.
Let U = $\cup_{a \in A} \beta^M_{\epsilon_a}(a)$ [same radius ϵ_a, but balls in M rather than in X $\Rightarrow$ these open sets are different – ones in M are "bigger".] Union of open balls around a in metric space M. Then U is open in M: union of open balls is open.
Claim: $A = X \cap U$. *Proof:* [prove both sides include the other]
Take any $a \in A$, then $a \in X$ since $X \subseteq A$, and $a \in U$ since $U \subseteq \beta^M_{\epsilon_a}(a)$. Thus, $A \subseteq X \cap U$.

Ex. M = $\mathbb{R}^2$, $X = \mathbb{R}$. Open ball in X is a circle (or more precisely, the diameter of this circle), open ball in M is a disk (inside of the circle is filled in.

Conversely, take any $y \in X \cap U$, then $y \in \beta^M_{\epsilon_a}(a)$ for some $a \in A$ since $y \in U = \cup_{a \in A} \beta^M_{\epsilon_a}(a)$. Thus, $d(a, y) < \epsilon_a$, but $y \in X$ [y is in then intersection of X and U], so $y \in \beta^X_{\epsilon_a}(a) \subseteq A$. Thus $X \cap U \subseteq A$.

Conversely, suppose that $A = X \cap U$ for some U that is open in (M,d). Take any $a \in A$, then $a \in U$ [A $= X \cap U$], then $\exists \epsilon$ such that $\beta^M_{\epsilon_a}(a) \subseteq U$. Take any $y \in X$ such that $d(a, y) < \epsilon$. Then $y \in \beta^M_{\epsilon_a}(a)$ [within distance ϵ of the bigger space and therefore in open ball] $\Rightarrow y \in U$. Thus, we conclude $\beta^X_{\epsilon_a}(a) \subseteq U$, but obviously $\beta^X_{\epsilon_a}(a) \subseteq X$, so $\beta^X_{\epsilon_a}(a) \subseteq X \cap U = A \Rightarrow A$ is open in (X,d).

Theorem 10.8 *$A \subseteq X$ is closed in (X,d) if and only if $A = X \cap F$ for some F that is closed in (M,d). Same result as for open sets.*

[Can try to prove this on your own – just take complements: open sets are complements of closed sets].

Theorem 10.9 *Let $X \subseteq M$, then $A \subseteq X$ is compact in the smaller space (X,d) if and only if A is compact in (M,d).*

Proof:
Suppose A is compact in (M,d). Take any sequence $\{x_n\}_{n=1}^{\infty}$in A. Then $\exists$ subsequence $\{x_{n_k}\}$ converging to some $x \in A$. But then this property also holds in (X, d). Existence of a convergent subsequence doesn't change if you move from a smaller sequence to a larger sequence. Compactness is an intrinsic property of A itself [doesn't change depending on which space the set is sitting in].Compactness is a property of the metric space itself – it doesn't depend on which space its sitting in.
Converse is similar.
In particular, A is a compact subset of (A,d) if and only if A is compact in (M,d).
Def. We say that a metric space (M,d) is compact if M is a compact subset of M. M is always an open set of M, M is always a compact set of M, but M is not necessarily a compact subset of itself. **Compactness is a function of the metric space**: (0,1) [open interval between 0,1] is not a compact metric space.

11 October 21

11.1 Continuous Functions

Let (M_1, d_2) and (M_2, d_2) be two metric spaces. Let f: $M_1 \to M_2$ be a function.
Def.[Continuity at a Point]
We say that f is continuous at a point $x \in M_1$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon$.

Effectively want to show $d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon$.

Ex. $f : \mathbb{R} \to \mathbb{R}$ (with Euclidean metric).
$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$.
Claim: f is continuous everywhere except at 0.

It's defined everywhere (so it's a well-defined function), and it's clear that it's continuous everywhere except at that point, but how do we prove this?

Take any $x \neq 0$. Take any $\epsilon > 0$. Suppose $|x - y| < \delta$ (with δ to be determined later).
$y \neq 0$, then $|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = \frac{|x-y|}{|x|*|y|} < \frac{\delta}{|x|*|y|} < \frac{\delta}{|x|*\frac{|x|}{2}} = \frac{2\delta}{x^2} < \epsilon$
Now suppose we chose δ so small $\delta < \frac{|x|}{2}$. Then $|x| \leq |x-y| + |y| < \delta + |y| < \frac{|x|}{2} + |y| \Rightarrow |y| > \frac{|x|}{2}$.
Then suppose also that $\delta < \epsilon \frac{x^2}{2}$. **Conclusion**: If we choose $\delta < \min\{\frac{|x|}{2}, \epsilon\frac{x^2}{2}\}$ then $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

For doing these types of proofs, do first part of calculation on a separate sheet of paper to find which δ you need to choose, and then start with "If we choose $\delta = ...$" when doing the actual proof.

What about not continuous at 0?
Discontinuity at 0: just have to exhibit one ϵ which works. Take $\epsilon = 1$.
To show: $\forall \delta > 0$, $\exists y$ such that $|0 - y| < \delta$ but $|f(0) - f(y)| \geq 1$.
Proof. Take any $\delta > 0$. Let $y = \min\{1, \frac{\delta}{2}\}$. Then $|0 - y| = |y| \leq \frac{\delta}{2} < \delta$, so y is within the delta ball around 0. But $|f(0) - f(y)| = |0 - \frac{1}{y}| = \frac{1}{|y|} \geq 1$ because $y \leq 1$. Therefore, we have discontinuity at 0.

Theorem 11.1 (Continuity on Metric Spaces) *f is continuous at a point x if and only if for any sequence $\{x_n\}_{n=1}^{\infty}$converging to x, $f(x_n)$ converges to $f(x)$.*

Proof
Suppose f is continuous at x. Take any sequence $\{x_n\}_{n=1}^{\infty}$converging to x. Take any $\epsilon > 0$, $\exists \delta > 0$ such that if $d_1(x,y) < \delta$, $\Rightarrow$ $d_2(f(x), f(y)) < \epsilon$. Since $\{x_n\}_{n=1}^{\infty} \to x$, $\exists N$ such that $\forall n \geq N$, $d_1(x_n, x) < \delta$. Then $\forall n \geq N$, $d_2(f(x_n), f(x)) < \epsilon$. Thus, $f(x_n) \to f(x)$.

Conversely, suppose that $\forall$ sequences $\{x_n\}_{n=1}^{\infty}$converging to x, we have that $f(x_n) \to f(x)$. WTS f is continuous at x.

Proof

Suppose not. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists y$ such that $d_1(x, y) < \delta$, but $d_2(f(x), f(y)) \geq \epsilon$ [means its not continuous].

Take any n, let $\delta = \frac{1}{n}$, then $\exists y_n$ such that $d_1(x, y_n) < \delta = \frac{1}{2}$, but $d_2(f(x), f(y_n)) \geq \epsilon$. And ϵ is fixed. but $y_n \to x$, but $f(y_n) \not\to f(x) \Rightarrow$ contradiction.

Def.[Continuous Function]
A function $f : M_1 \to M_2$ is called continuous if it is continuous at every $x \in M_1$.

Relation between Continuity and Compactness

Theorem 11.2 *Suppose $X \subseteq M_1$ is compact and $f : M_1 \to M_2$ is continuous. Then the image of f [i.e. f(x) = $\{f(x)|x \in X\}$] is a compact subset of M_2.*

main point: Continuous functions map compact sets to compact sets.
Proof:
Take any sequence $\{y_n\}_{n=1}^{\infty}$ in $f(X)$. Then $\forall n$, $y_n = f(x_n)$ for some $x_n \in X$. By compactness of X, $\exists$ subsequence $\{x_{n_k}\}$ converging to some $x \in X$. Then by continuity of f, $y_{n_k} = f(x_{n_k}) \to f(x) = y \in f(X)$.

Corollary 11.2.1 *Suppose that (M, d) is a compact metric space, and $f : M \to \mathbb{R}$ is a real-valued function. Then f is a bounded function, meaning that $\exists R \geq 0$ such that $|f(x)| \leq R$ $\forall x \in M$.*

Proof:
Compact in the real line implies closed and bounded.
Since M is compact and f is continuous, the set $f(M) \subset \mathbb{R}$ is also compact, and hence closed and bounded. Therefore f is a bounded function.

Ex.
Consider M = (0,1). Suppose $f : M \to \mathbb{R}$ so that $f(x) = \frac{1}{x}$. Then f is continuous on M, but f is clearly unbounded. But our theorem implies that such a function can never be constructed on the closed interval [0,1]. If your function is unbounded, it must have a point of discontinuity somewhere.

Brower-fixed point theorem: any function from disc to a disc or ball into a ball will have a fixed point. Here's how Brower proved it.
B = disc and suppose we have a function $f : B \to B$, and suppose there is no fixed point: no x such that f(x) = x.

12 October 24

12.1 Continuous Functions

One of the main results: Image of a compact set is also compact.
Midterm: up to last Friday's lecture.
Similar standard to the practice midterm.

Theorem 12.1 *Let (M_1, d_1) AND (M_2, d_2) be metric spaces.*
Let $f : M_1 \to M_2$ be a function. Then the following are equivalent:
1. f is continuous.
2. $\forall C \subseteq M_2$ closed, $f^{-1}(C) = \{x \in M_1 | f(x) \in C\}$ is also closed [inverse image of a closed set is closed].
3. $\forall U \subseteq M_2$ open, $f^{-1}(U)$ is open.

Useful for general topological spaces when you would like to define continuity. But on metric spaces, these are equivalent notions.
Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). (1) $\Rightarrow$ (2)
Let $C \subseteq M_2$ be closed. Take a sequence $\{x_n\}_{n=1}^{\infty}$in $f^{-1}(C)$ converging to some $x \in M_1$. WTS: $x \in f^{-1}(C)$. Since $x_n \in f^{-1}(C)$, then $f(x_n) \in C$. Now $x_n \to x$, and f is continuous, so $f(x_n) \to f(x)$, but $f(x_n) \in C \ \forall n$ and C is closed. Thus, $f(x) \in C$ which means that $x \in f^{-1}(C)$. Therefore $f^{-1}(C)$ is closed.

(2) $\Rightarrow$ (3)
Proving $f^{-1}(U^C) = f^{-1}(U)^C$ will help us show (2) $\Rightarrow$ (3).

Let $U \subseteq M_2$ be open. Then U^C is closed. By (2), $f^{-1}(U^C)$ is closed. Now if $x \in f^{-1}(U^C)$ then $f(x) \in U^C$ and so $f(x) \notin U$. Thus $x \notin f^{-1}(U)$ which shows that $x \in f^{-1}(U)^C$. So $f^{-1}(U^C) \subseteq f^{-1}(U)^C$.

On the other hand, if $x \in f^{-1}(U)^C$, then $f(x) \notin U$ and so $f(x) \in U^C$ and thus $x \in f^{-1}(U^C)$.
So $f^{-1}(U)^C = f^{-1}(U^C)$ is closed. $\Rightarrow f^{-1}(U)$ is open. (3) $\Rightarrow$ (1)
Want to show f is continuous at every point in M_1.
Take any $x \in M_1$ and take any $\epsilon > 0$
To Show: $\exists \delta > 0$ such that $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon$. Have to prove that there exists such a delta.

Let $U = \beta_\epsilon(f(x)) \subseteq M_2$. Then U is open. Also, $f(x) \in \beta_\epsilon(f(x))$, so $x \in f^{-1}(U)$. But by (3) $f^{-1}(U)$ is open since U is open. Thus, $\exists \delta > 0$ such that $\beta_\delta(x) \subseteq f^{-1}(U)$.
Take any $y \in M_1$ such that $d_1(x, y) < \delta$. Then by definition, $y \in \beta_\delta(x) \subseteq f^{-1}(U)$. Thus $f(y) \in U$, and so $d_2(f(x), f(y)) < \epsilon$.

12.2 Uniform Continuity

Def A function $f : M_1 \to M_2$ is called uniformly continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that whenever $x, y \in M_1$ with $d_1(x, y) < \delta$, we have $d_2(f(x), f(y)) < \epsilon$.

In continuity δ depends on x. In this one, it only depends on ϵ.

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not uniformly continuous. $f(x) - f(y) = x^2 - y^2 = (x - y)(x + 1)$.

Take any $\delta > 0$, let $y = x + \delta$, $x > 0$. Then $f(x) - f(y) = \delta(2x + \delta) > 1$ if x is large enough.

Function $f(x) = x$ is uniformly continuous in $\mathbb{R}$. Can choose $\delta = \epsilon$ and you're done for this example.

$f : (0, \frac{\pi}{2} \to \mathbb{R}$, $f(x) = tan(x)$ not uniformly continuous.
$f : (0, 1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ not uniformly continuous.

$l^1 = \{(x_1, x_2, ...)| \sum\limits_{n=1}^{\infty} |x_i| < \infty\}$ and $f(x) = x_1^2$. $d(x, y) = \sum\limits_{n=1}^{\infty} |x_i - y_i|$. Take sequence $(x, 0, 0, ...)$ and $(x + \delta, 0, 0, ...)$ differ by δ but $f(x) - f(y)$ can become arbitrarily large.

Bounded (and closed subset of l^1 which is not compact): $X = \{x \in l^1 | \sum\limits_{n=1}^{\infty} |x_i| \leq 1\}$ since $(0, 0, 0, ..., 1, 0, 0, ...)$ cannot be cauchy.

Theorem 12.2 *Let M_1 be a compact metric space and M_2 be any metric space. Let $f : M_1 \to M_2$ be continuous. Then f is uniformly continuous.*

Proof:
Take any $\epsilon > 0$. Take any $x \in M_1$. There exists $\delta_x > 0$ such that $d_1(x, y) < \delta_x \Rightarrow d_2(f(x), f(y)) < \frac{\epsilon}{2}$ [by continuity of f]. Thus, $\beta_{\delta_x}(x) \subseteq f^{-1}(\beta_{\frac{\epsilon}{2}}(f(x))$.
Now $\{\beta_{\frac{\delta_x}{2}}(x)\}_{x \in M_1}$ is an open cover of M_1.
So by compactness there is a finite subcover: $\{\beta_{\frac{\delta_{xi}}{2}}(x_i)\}_{i=1}^n$.
Let $\delta = \min\{\frac{\delta x_1}{2}, .., \frac{\delta x_n}{2}\} > 0$. Take any $x, y \in M_1$ such that $d_1(x, y) < \delta$. $\exists i \in \{1, 2, ..., n\}$ such that $x \in \beta_{\frac{\delta_{xi}}{2}}(x_i)$ [these sets form an open cover of M_1 therefore there exists some i].
Therefore, $d_1(x, x_i) < \frac{\delta_{x_i}}{2}$. So, $d_1(y, x_i) \leq d_1(y, x) + d_1(x, x_i) < \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i}$
Thus $x, y \in \beta_{\frac{\delta_{xi}}{2}}(x_i)$
$\Rightarrow d_2(f(x), f(x_i)) < \frac{\epsilon}{2}$ and $d_2(f(y), f(x_i)) < \frac{\epsilon}{2}$. $\Rightarrow d_2(f(x), f(y)) < \epsilon$.

13 October 26

13.1 Connected Metric Spaces

Def. A metric space is (M,d) is said to be connected if the only subsets of M that are both open and closed are $\emptyset$ and M.

Theorem 13.1 *M is connected* $\iff$ $\not\exists$ *disjoint non-empty open subsets* $U, V \subseteq M$ *such that* $M = U \cup V$ *[M cannot be written as two disjoint non-empty subsets].*

Proof: Suppose M is connected. If such U, V exist, then they are both open and closed [Complements are also open since $U^C \cup V^C$ is equal to M and therefore complements are open, therefore U and V are both closed and we assumed they were open...], and neither $= \emptyset$ or M.
Conversely, suppose such U, V do not exist. Let $U \subseteq M$ be a set that is both open and closed. Then U^C is also open, and either U is empty or U^C is empty.
Def. Let (M,d) be a metric space. A subset $X \subseteq M$ is called connected if (X, d) is a connected metric space [i.e. X is connected in the relative metric].

Theorem 13.2 *A subset* $X \subseteq \mathbb{R}$ *[usual metric on* $\mathbb{R}$*] is connected if and only if whenever* $a, b \in X$*, and* $a < b$*, we have* $[a, b] \subseteq X$*. Subset of the real line is connected if and only if it has this property.*

Proof:
Suppose X is connected. Suppose that X does not have the above property. That is, $\exists a, b \in X$, $a < b$ such that $[a, b] \not\subseteq X$. Then $\exists c \in [a, b]$ such that $c \notin X$. Let $U = (-\infty, c)$, $V = (c, \infty)$, then U and V are open subsets of $\mathbb{R}$, and so $X \cap U$ and $X \cap V$ are open subsets of X. Both are non-empty, since $a \in X \cap U$ and $b \in X \cap V$. And finally, $X = (X \cap U) \cup (X \cap V)$ since $X \subseteq (X \cap U) \cup (X \cap V)$ is obvious and for any $x \in X$, we have $x \neq c$ (since $c \notin X$), so $x \in U \cup V$ which shows that $(X \cap U) \cup (X \cap V) \subseteq X$. This contradicts the statement that X is connected.

Conversely, suppose X has the given property. Suppose X is not connected. Then, $\exists$ open sets U, V $\subseteq \mathbb{R}$ such that $X \cap U$, $X \cap V$ are non-empty and $X = (X \cap U) \cup (X \cap V)$ and $(X \cap U) \cap (X \cap V) = \emptyset$ [X can be decomposed into two disjoint, non-empty subsets, but U and V need not be disjoint on $\mathbb{R}$ – scope is X, not $\mathbb{R}$].

Take $a \in X \cap U$, $b \in X \cap V$, without loss of generality, let's suppose that $a < b$ [$a \neq b$ because disjoint sets]. Let $c = \sup\{x \in [a, b] | x \in X \cap U\}$ [this set is non-empty since $a \in$ this set, and it's bounded because its a subset of $[a, b]$ $\Rightarrow$ supremum is well defined]. Call this set A. Note that $c \in X$ since X has the given property: $[a, b] \subseteq X$.

Then (1) For any n, $\exists x_n \in A$ so that $x_n > c - \frac{1}{n}$. Obviously, $x_n \leq c$, thus $x_n \to c$. So $x_n \in X \cap U \; \forall n$, $x_n \to c$.
Didn't prove, but it's a property: if you take the supremum of a set, there's a convergent subsequence to the supremum.
But $X \cap U$ is a closed subset of X [complement of $X \cap U$ is $X \cap V$ is open in X], and $c \in X$, so $c \in X \cap U$. But $X \cap U$ is open in X. So $\exists$ some $\epsilon > 0$ such that $\{y \in X || c - y| < \epsilon\}$ is contained within $X \cap U$ [$X \cap U$ is open in X].
Note: $c < b$ since $c \in [a, b]$ and $c \neq b$ because $b \in X \cap V$. Choose $\epsilon < b - c$ smaller than ϵ [can make ϵ as small as we want], then $c + \frac{\epsilon}{2} < b$. Thus $c + \frac{\epsilon}{2} \in [a, b] \subseteq X$ but by $\{y \in X || c - y| < \epsilon\} \subseteq X \cap U$, $c + \frac{\epsilon}{2} \in X \cap U$. But this contradicts the definition of c [found a point slightly bigger than c in $X \cap U$].

Corollary 13.2.1 *A set $X \subseteq \mathbb{R}$ is connected if and only if X is an interval (may be open, closed, half-open, bounded, or unbounded).*

Proof:
Let a = $\inf X$, b = $\sup X$, then by the given property, you can show that X must be either [a,b] or (a,b) or [a,b) or (b,a] (works for ∞, $-\infty$ as well).

Corollary 13.2.2 *The real line is connected.*

Corollary 13.2.3 *The real space $\mathbb{R}^n$ is connected.*

14 October 31

14.1 Connectedness

Theorem 14.1 *Let (M_1, d_1) and (M_2, d_2) be metric spaces. Let $f : M_1 \to M_2$ be a continuous function. Suppose M_1 is connected. Then $f(M_1)$ is also connected.*

Property of connectedness is intrinsically a property of the metric space [like compactness]. Doesn't really matter if you take a subset of it. *Proof:* [Have to verify every pair of disjoint sets doesn't complete the space]
Suppose not. $\exists$ disjoint sets $U, V \subseteq f(M_1)$ such that both U, V are nonempty and open in $f(M_1)$ and $f(M_1) = U \cup V$.

Note: If we consider $f : M_1 \to f(M_1)$ is continuous. Thus, $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of M_1 (inverse image of open maps is open). Both are non-empty since U and V are non-empty and are contained in $f(M_1)$.

Claim: $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint.
If $x \in f^{-1}(U) \cap f^{-1}(V)$, then $f(x) \in U$ and $f(x) \in V$ which is impossible since U and V are disjoint (non-empty, and open).

Claim: $f^{-1}(U) \cup f^{-1}(V) = M_1$
It is obvious that $f^{-1}(U) \cup f^{-1}(V) \subseteq M_1$. For the opposite conclusion, take

any $x \in M_1$. Then $f(x) \in f(M_1) = U \cup V$. So $f(x) \in U$ or $f(x) \in V$ $\rightarrow x \in f^{-1}(U) \cup f^{-1}(V)$.

This gives us a contradiction since it cannot be in the union of two disjoint non-empty subsets.

Def. A continuous path in a metric space (M, d) is a continuous map $f : [0, 1] \rightarrow M$. (Sometimes, the image of $[0, 1]$ under f is called the "path").

14.2 Path Connectedness

Def. A Metric space (M,d) is called "path connected" if for any $x, y \in M$, there is a continuous path $f : [0, 1] \rightarrow M$ such that $f(0) \rightarrow x$, $f(1) \rightarrow y$.

Theorem 14.2 *A path connected metric space is connected.*

Proof:
Suppose that M is path-connected but not connected. Let U, V be disjoint non-empty, open subsets of M such that $U \cup V = M$. Since U,V are non-empty, $\exists$ $x \in U$, $y \in V$. Since M is path connected, $\exists$ a continuous map $f : [0, 1] \rightarrow M$ such that $f(0) = x$ and $f(1) = y$. Now let $X = f([0, 1])$ (the image of [0,1] under f: the path itself). Since $[0, 1]$ is connected and f is continuous, X is connected. So let $U' = X \cap U$ and $V' = X \cap V$. Then:

- $U' \cap V'$ are disjoint since $U \cap V = \emptyset$
- U' and V' are open in X [X is a metric space under d, X intersect open subset of M must be open]
- $U' \cup V' = X$ since $U \cup V = M$.
- U', V' are non-empty since $x \in U'$ and $y \in V'$.

This shows that X is not connected, and therefore, we have a contradiction.

Corollary 14.2.1 $\mathbb{R}^d$ *is connected for any d.*

Proof:
Take any $x, y \in \mathbb{R}^d$. Define $f : [0, 1] \rightarrow \mathbb{R}^d$ as $f(t) = (1 - t)x + ty$. This is a continuous map with $f(0) = x$ and $f(1) = y$ [straight line connecting x and y].
Application:
Consider $\mathbb{R}^3$ with $S^2 \subseteq \mathbb{R}^3$ where $S = \{x \in \mathbb{R}^3 : |x| = 1\}$ where $|x|$ is the unit norm $\sqrt{x_1^2 + x_2^2 + x_3^2}$. This is the unit sphere in $\mathbb{R}^3$. This is a closed and bounded set (and hence compact).

Let $f : S^2 \rightarrow \mathbb{R}$. be a continuous function. (Think of S^2 as the surface of the earth and $f(x)$ to be the temperature at x).

Ex. There exists $x \in S^2$ such that $f(x) = f(-x)$.
Proof:
Define $g(x) = f(x) - f(-x)$: The difference between the temperature at a point and its antipodal point ($f(-x)$). Note $f(-x) = f(h(x))$ where $h(x) = -x$. f, h are both continuous maps, thus $f \circ h$ is also continuous [can prove easily]. Thus, g is continuous. Take any $x \in S^2$. If $g(x) = 0$, there is nothing to prove.
Suppose $g(x) \neq 0$. Note: $g(-x) = -g(x)$. Thus, $g(x)$ and $g(-x)$ have opposite signs. But since S^2 is path-connected, and hence connected, $g(S^2)$ is also connected. But $g(S^2) \subseteq \mathbb{R}$. Thus $g(S^2)$ in an interval. $\Rightarrow 0 \in g(S^2)$ so we're done.

Theorem 14.3 (Intermediate Value Theorem) *Let (M, d) be a connected metric space. Let $f : M \to \mathbb{R}$ be a continuous function. If $f(x) = a$ and $f(y) = b$ for some $x, y \in M$ and $a, b \in \mathbb{R}$, then $[a, b] \subseteq f(M)$ ($a \leq b$).*

Proof:
$f(M)$ is connected, so the whole interval between the two points are contained within the image of the function.

15 November 2nd

15.1 Complete Metric Spaces

Def. A metric space (M,d) is said to be complete if every Cauchy Sequence in M converges to some point in M.

Ex. $\mathbb{R}$ with the usual metric is complete [use least upper bound to prove monotone sequences converge, BW to prove that bounded seq has a subsequence, etc. – took a lot of work].
Ex. $\mathbb{Q}$ with the usual metric is not complete.
Why? We proved that $\sqrt{2}$ is not a rational number. We also know that $\forall n$, $\exists$ a rational number $x_n \in [\sqrt{2} - \frac{-1}{n}, \sqrt{2}]$ Then $\{x_n\}_{n=1}^{\infty}$ is Cauchy and $\lim\limits_{n\to\infty} x_n = \sqrt{2} \notin \mathbb{Q}$. Just have to exhibit one sequence that is Cauchy but not convergent to show that it is not complete.

Theorem 15.1 *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in any metric space (M,d), then $\{x_n\}_{n=1}^{\infty}$ is Cauchy.*

Proof: Suppose $\{x_n\}_{n=1}^{\infty} \to x$. Take any $\epsilon > 0$. Find N such that $\forall n \geq N$, $d(x_n, x) < \frac{\epsilon}{2}$. Then $\forall m, n \geq N$, $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \epsilon$.
Ex $\mathbb{R}^n$ is complete. *Proof:* If $\{x_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}^n$, then each coordinate is a Cauchy sequence in $\mathbb{R}$, and so convergent. And thus, $\{x_n\}_{n=1}^{\infty}$ converges. Distance between two coordinates can become arbitrarily small $\Rightarrow$ the total distance between two elements of $\mathbb{R}^n$ becomes arbitrarily small.

Theorem 15.2 *l^1 is complete.*

Proof:
Let $\{a^{(n)}\}_{n=1}^{\infty}$ be a Cauchy sequence in l^1. Let $a^{(n)} = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, ...)$. Each coordinate will be a Cauchy sequence in the real numbers. Coordinate-wise convergence does not imply convergence in l^1, but proving coordinate-wise convergence is not difficult in l^1.
Step 1: For each i, $\{a_i^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers.

Proof: Take any $\epsilon > 0$. Find N such that $\forall n, m \geq N$, $d(a^{(m)}, a^{(n)}) < \epsilon$. Take any coordinate i. Then $\forall m, n \geq N$, $|a_i^{(m)} - a_i^{(n)}| \leq \sum_{j=1}^{\infty} |a_j^{(m)} - a_j^{(n)}| = d(a^{(m)}, a^{(n)}) < \epsilon$. This proves that for each coordinate, you have a cauchy sequence of real numbers that converges.

Thus, $\forall i \; a_i = \lim\limits_{n\to\infty} a_i^{(n)}$ exists. (coordinate-wise, the limit exists).

Let $a = (a_1, a_2, a_3, ...)$. Need to show $a \in l^1$. And $a_n \to a$.

Step 2: $a \in l^1$.
Proof: Take any $\epsilon > 0$. Let N be such that $\forall m, n \geq N$, $d(a^{(m)}, a^{(n)}) < \epsilon$. Then $\forall n \geq N$, $\sum_{i=1}^{\infty} \leq \sum_{i=1}^{\infty} (|a_i^{(n)} - a_i^{(N)}| + |a_i^{(N)}| = \sum_{i=1}^{\infty} |a_i^{(n)} - a_i^{(N)}| + \sum_{i=1}^{\infty} |a_i^{(N)}| < \epsilon + \sum_{i=1}^{\infty} |a_i^{(N)}$

Trick: Take any $p \geq 1$. Then $\forall n \geq N$, $\sum_{i=1}^{p} |a_i^{(n)}| \leq \sum_{i=1}^{\infty} |a^{(n)}| \leq \epsilon + \sum_{i=1}^{\infty} |a_i^{(N)}|$

We know $\lim\limits_{n\to\infty} \sum_{i=1}^{p} |a_i^{(n)}| = \sum_{i=1}^{p} \lim\limits_{n\to\infty} |a_i^{(n)}| = \sum_{i=1}^{p} |a_i|$ [can exchange limit and sum for finite sum].

Thus, $\forall p$, $\sum_{i=1}^{p} |a_i| \leq \epsilon + \sum_{i=1}^{\infty} |a_i^{(N)}|$ [does not depend on p]. Thus, we can now take $p \to \infty$ on the left and get $\sum_{i=1}^{\infty} |a_i| \leq \epsilon + \sum_{i=1}^{\infty} |a_i^{(N)}| < \infty$. So $a \in l^1$.

Step 3: $a^{(n)} \to a$ in l^1.
Proof: Take any $p \geq 1$. Take any ϵ. Find N such that $\forall m, n \geq N$, $d(a^{(m)}, a^{(n)}) < \epsilon$. Then $\sum_{i=1}^{P} |a_i^{(m)} - a_i^{(n)}| \leq \sum_{i=1}^{\infty} |a_i^{(m)} - a_i^{(n)}| = d(a^{(m)}, a^{(n)}) < \epsilon$ (we call this *).

Now fixing $n \geq N$, send $m \to \infty$ in (*). We get $\sum_{i=1}^{P} |a_i - a_i^{(n)}| < \epsilon$. But this holds $\forall p$. So we conclude that w can send $p \to \infty$. $d(a, a^{(n)}) = \sum_{i=1}^{\infty} |a_i - a_i^{(n)}| \leq \epsilon$. Therefore, l^1 is complete.

Def. Let (M,d) be a metric space. We say that (M_1, d_1) is a completion of (M,d) if the following three conditions hold:
1) (M_1, d_1) is complete

2) $M \subseteq M_1$
3) $d(x, y) = d_1(x, y)$ if $x, y \in M$.
Enlarge the metric space M with more points so that it becomes complete.

Theorem 15.3 *Every metric space has a completion.*

Equivalence classes of Cauchy Sequences.
Proof: Define two Cauchy Sequences in M to be equivalent if $d(x_n, y_n) \to 0$ as $n \to \infty$.
Let M_1 equal the set of equivalence classes. Define $d_1(\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}) = \lim_{n\to\infty} d(x_n, y_n)$ [and can prove this limit always exists by the triangle inequality].
Read from the textbook...
Map from $M \to M_1$ such that we can repeat the element in M infinitely many times to fall into an equivalence class of M_1. Need to show every Cauchy Sequence in M_1 has a limit.

16 November 7

16.1 Complete Metric Spaces

Def. Pointwise convergence
Let M be a metric space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from X into $\mathbb{R}$. We say $f_n \to f$ pointwise if for each $x \in M$, $\lim_{n\to\infty} f_n(x) = f(x)$.
Fact Even if each f_n is continuous and $f_n \to f$ pointwise, f may not be continuous.
Ex. Let $M = [0, 1]$. Let $f_n(x) = x^n$. Then $f_n \to f$ pointwise where f(x) is 1 if x = 1, and 0 if $x \in [0, 1)$. But f is not a continuous function (discontinuity at 1), even though f converges pointwise.

Pointwise convergence: sequence of functions where each function in this sequence converges (not each point in one specific function).

Def. We say that $f_n \to f$ uniformly [stronger than pointwise] if $\forall \epsilon > 0$, $\exists N \geq 1$ such that $\forall n \geq N$, $\forall x \in M$, $|f_n(x) - f(x)| < \epsilon$. [This one N works for all x – in pointwise convergence N depends on x]. Same N works for all x.

Theorem 16.1 *If each f_n is continuous at a point $x \in M$, and $f_n \to f$ uniformly, then f is continuous at x. In particular, if each f_n is continuous everywhere, then f is continuous everywhere.*

Proof: Take any $\epsilon > 0$. Find N so large that $\forall n \geq N$, $\forall y \in M$, $|f_n(y) - f(y)| < \frac{\epsilon}{3}$. Since f_N is continuous at x, $\exists \delta > 0$ such that if $d(x, y) < \delta$, then $|f_N(y) - f_N(x)| < \frac{\epsilon}{3}$ (because f_N is continuous at x).
Claim: If $d(x, y) < \delta$, then $|f(x) - f(y)| < \epsilon$ (will prove f is continuous at x).

Proof: Take any $y \in \beta_\delta(x)$. Then $|f(x) - f)y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + f_N(y) - f)y)|$ where each of these terms are bounded by $\frac{\epsilon}{3}$, so $\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. If only pointwise convergence, wouldn't have that the third term is bounded by $\frac{\epsilon}{3}$ (have it for the first two, but not the third).

16.2 Function Spaces

Let M be a compact metric space. Let C(M) denote the space of all continuous functions from M into $\mathbb{R}$. [Only work with compact metric spaces]. For $f, g \in C(M)$, let $\rho(f,g) = \sup_{x \in M} |f(x) - g(x)|$ [sometimes called the supremum metric].
Claim: ρ is a metric on C(M).
Proof: For any $f, g \in C(M)$, $\rho(f,g) \geq 0$ [clear], and $\rho(f,g)$ is finite since f, g being continous functions on a compact space are bounded. If you don't have a compact space, everything will go through, but the distance will sometimes be ∞ (but distance needs to be finite for the definition of a metric).

So $\rho(f,g) = \rho(g,f)$ is also clear.
$\rho(f,g) = 0 \iff f = g$ is also clear.
Take $f, g, h \in C(M)$, then $\rho(f,g) = \sup\{|f(x) - g(x)| x \in M\} \leq \sup\{|f(x) - h(x)| + |h(x) - g(x)| : x \in M\}$ (via triangle ineaquality). $\leq \sup\{|f(x) - h(x) : x \in M\} + \sup\{|h(x) - g(x)| : x \in M\} = \rho(f,h) + \rho(h,g)$.

Theorem 16.2 *$f_n \to f$ in C(M) if and only if $f_n \to f$ uniformly.*

Proof: Suppose that $f_n \to f$ in $C(M)$, that is $\rho(f_n, f) \to 0$. Take any $\epsilon > 0$. Then $\exists N$ such that $\forall n \geq n$, $\rho(f_n, f) < \epsilon$ which implies $\forall n \geq N$, $\sup_x |f_n(x) - f(x)| < \epsilon \Rightarrow \forall n \geq N, \forall x \in M$, $|f_n(x) - f(x)| < \epsilon$ [convergence w.r.t. $\rho \Rightarrow$ uniform convergence].
Conversely, suppose that $f_n \to f$ uniformly. Then Take any $\epsilon > 0$. Find N such that $\forall n \geq N$, and $\forall x \in M$, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ [by uniform convergence]. Thus, $\forall n \geq N$, $\rho(f_n, f) = \sup_x |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$. Supremum gives us $\leq$, but we want $< \epsilon$.

Theorem 16.3 *$(C(M), \rho)$ is a complete metric space.*

This space comes up in Math 230C (last course in the probability sequence). Brownian motion is random continuous function that is not differentiable anywhere. Very important space (space of continuous functions of a compact set)
Proof: Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $C(M)$. Take any $x \in M$, then $|f_n(x) - f_m(x)| \leq \sup_{y \in M} |f_n(y) - f_m(y)| = \rho(f_m, f_n)$. From the fact that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence, $\{f(x)_n\}_{n=1}^\infty$ is a Cauchy sequence of real numbers and therefore has a limit. Let's call the limit $f(x)$.
Need to show $f(x) \in C(M)$ and $\{f(x)_n\}_{n=1}^\infty$ converges to $f(x)$ in the ρ metric. Take any $\epsilon > 0$. Find N such that $\forall n, m \geq N$, $\rho(f_m, f_n) < \epsilon$. Thus, $\forall x \in M$,

$\forall n, m \geq N,\ |f_m(x) - f_n(x)| < \epsilon \Rightarrow \forall x \in M,\ \forall n \geq N,\ |f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \epsilon$. This shows that $f_n \to f$ uniformly. $\Rightarrow f \in C(M)$ (f is continuous) and $\rho(f_n, f) \to 0$ [shows both of the things we wanted to show]. **Have that (f(x) $\to \mathbb{R}$) as this function maps points to** $\mathbb{R}$. This space is complete, but not compact. Any compact metric space is complete [any Cauchy sequence has a convergent subsequence, therefore it converges to the convergent subsequence, therefore all Cauchy sequences converge, therefore space is complete].

16.3 Riemann Integration

Let $[a, b]$ be a closed interval. A partition of [a,b] is a finite set $P = \{x_0, ..., x_n\}$, with $a = x_0 < x_1 < x_2 < ... < x_n = b$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function (upper bound and lower bound on the values of f). We define the upper and lower sums of f with respect to the partition P as $U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, $L(P, f) = \sum_{n=1}^{\infty} m_i(x_i - x_{i-1})$ where $M_i = \sup\{f(x)|x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ [upper and lower bound each interval].

17 November 09

17.1 Riemann Integration

$f : [a, b] \to \mathbb{R}$
f is bounded, meaning that $\exists M \geq 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.
Take a partition $P = \{x_0, x_1, ..., x_n\}$, $a = x_0 < x_1 < ... < x_n = b$. $U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, $M_i = \sup\{f(x)|x \in [x_{i-1}, x_i]\}$. $U(f, p)$ is a reasonable upper bound for the area under the curve.
Similarly, the lower Riemann sum: $L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$, $m_i = \inf\{f(x)|x \in [x_{i-1}, x_i]\}$. This is indeed a lower bound on the area under the curve.

Observe that for any partition P, $L(f, P) \leq U(f, P)$ (since $m_i \leq M_i\ \forall i$).
Proof: Let P be a partition and $P' = P \cup \{x\}$ be a partition with one additional point $x \notin P$.
Claim: $U(f, P') \leq U(f, P)$ and $L(f, P') \geq L(f, P)$.
Proof: Suppose $x_{i-1} < x < x_i$. Then $U(f, P') = M_1(x_1 - x_0) + ... + M_{i-1}(x_{i-1}, -x_{i-2})$ $= M_i^{(1)}(x - x_{i-1}) + M_i^{(2)}(x_i - x) + M_{i+1}(x_{i+1} - x_i) + ... + M_n(x_n - x_{n-1})$.
$M_i^{(1)} = \sup f(y) \leq M_i$ such that $y \in [x_{i-1}, x]$
$M_i^{(2)} = \sup f(y) \leq M_i$ such that $y \in [x, x_i]$

$\leq M_i(x_1 - x_2) + ... + M_{i-1}(x_{i-1} - x_{i-2}) + +M_{i+1}(x_{i+1} - x_i) + M_i(x_i - x) +$

$... + M_n(x_n - x_{n-1})$
So we have $M_i(x - x_{i-1}) + M_i(x_i - x) = M_i(x_i - x_{i-1})$
$U(f, P') \le U(f, P)$, Similarly, $L(f, P') \ge L(f, P)$.
A partition S is called a refinement of P if $P \subseteq S$. By using the previous claim and induction, we get that if S is a refinement of P, then $U(f, S) \le U(f, P)$, and $L(f, S) \ge L(f, P)$.

Lemma 17.1 *For any two partitions P and S, $L(f, S) \le U(f, P)$.*

Proof: Note that $P \cup S$ is a refinement of both P and S [a common refinement]. Thus, $L(f, S) \le L(f, P \cup S) \le U(f, P \cup S) \le U(f, P)$.

Let $\mathcal{L} = \{L(f, P) | P \text{ is a partition of } [a, b]\}$. (f, a, b are fixed).
$\mathcal{U} = \{U(f, P) | P$ is a partition of [a, b]$\}$.

- Define the lower Riemann Integral of f as $\int_{\underline{a}}^{b} f(x)dx = \sup \mathcal{L}$
- And the upper Riemann integral $\int_{a}^{\overline{b}} f(x)dx = \inf \mathcal{U}$.

We say that f is Riemann integrable if $\int_{\underline{a}}^{b} f(x)dx = \int_{a}^{\overline{b}} f(x)dx$ and then this is defined to be $\int_a^b f(x)dx$.
Note that $\forall$ bounded f, $\int_{\underline{a}}^{b} f(x)dx \le \int_{a}^{\overline{b}} f(x)dx$.

Proof: For any P, $U(f, P) \ge L(f, S)$ $\forall S$. This implies $U(f, P) \ge \int_{\underline{a}}^{b} f(x)dx$. This implies $\int_{a}^{\overline{b}} f(x)dx \ge \int_{\underline{a}}^{b} f(x)dx$.

17.2 Riemann's Criterion

When Riemann integrals exist (i.e. lower integral is equal to upper integral). Let $\mathcal{R}[a, b]$ denote the set of all Riemann integrable functions on [a,b].

Theorem 17.2 *Riemann's Criterion: $f \in \mathcal{R}[a, b]$ if and only if $\forall \epsilon > 0$, $\exists P$ such that $U(f, P) - L(f, P) < \epsilon$.*

Proof: Suppose that $f \in \mathcal{R}[a, b]$. Take any $\epsilon > 0$. Since $\int_{\underline{a}}^{b} f dx = \sup \mathcal{L}$, $\exists P$ such that $L(f, P) \ge \int_{\underline{a}}^{b} f dx - \frac{\epsilon}{2}$ [by the definition of the supremum]. And similarly, $\exists S$ such that $U(f, S) < \int_{a}^{\overline{b}} f dx + \frac{\epsilon}{2}$ [by the definition of the infimum]. Since $f \in \mathcal{R}[a, b]$, $\int_{\underline{a}}^{b} f dx = \int_{a}^{\overline{b}} f dx$.
So $U(f, S) - L(f, P) < \int_a^b f dx + \frac{\epsilon}{2} - (\int_a^b f dx - \frac{\epsilon}{2} = \epsilon$ [requires same partition S and P for Riemann Integrable $\Rightarrow$ take Refinement].
Thus, $U(f, P \cup S) - L(f, P \cup S) \le U(f, S) - L(f, P) < \epsilon$.

Now suppose there exists such a partition. Suppose that $\forall \epsilon > 0$, $\exists P$ such that $U(f,P) - L(f,P) < \epsilon$. Take any $\epsilon > 0$. Find such a P. Then $\overline{\int_a^b} f dx \leq U(f,P)$ and $\underline{\int_a^b} f dx \geq L(f,P)$. Thus, $\overline{\int_a^b} f dx - \underline{\int_a^b} f dx \leq U(f,P) - L(f,P) < \epsilon$ [by the definition of the upper integral and lower integral]. Thus, $\overline{\int_a^b} f dx - \underline{\int_a^b} f dx \leq 0$, but we also know that this is ≥ 0. Therefore, it must be 0.

Theorem 17.3 $C[a,b] \subseteq \mathcal{R}[a,b]$

C is continuous function. Take any $f \in C[a,b]$. By the continuity of f and the compactness of $[a,b]$, f is bounded. Also by compactness of [a,b], f is uniformly continuous. Take any $\epsilon > 0$. Find $\delta > 0$ so small that $|x-y| < \delta \to |f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$ [by the definition of uniform continuity] for any $x, y \in [a,b]$.

Let $P = \{x_0, ..., x_n\}$ be a partition of $[a,b]$ such that $x_i - x_{i-1} < \delta$ $\forall i$. Then for any i, $M_i - m_i = \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x)$. Need to show supremum - infimum is bounded by $\frac{\epsilon}{2(b-a)}$

18 November 11

18.1 Riemann Integrable

Theorem 18.1 *Any continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof: Since f is continuous and [a,b] is compact, f is bounded and uniformly continuous. Take any $\epsilon > 0$, find $\delta > 0$ so small that for all $x, y \in [a,b]$ such that $|x-y| < \delta$, we have that $|f(x) - f(y)| < \frac{\epsilon}{(b-a)}$ [by uniform continuity]. Let P $= \{x_0, ..., x_n\}$ be a partition of [a,b] such that $(x_i - x_{i-1}) < \delta \forall i$.

Let $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. Both are finite since f is bounded. In the last lecture, we claimed the difference between the two is bounded by $\frac{\epsilon}{(b-a)}$.

Fix some i. Take any $\theta > 0$. $\exists w \in [x_{i-1}, x_i]$ so that $f(w) > M_i - \theta$ [anything a little less than supremum is in the interval]. $\exists z \in [x_{i-1}, x_i]$ such that $f(z) < m_i + \theta$ [by the definition of the infimum]. Since $w, z \in [x_{i-1}, x_i]$ and $x_i - x_{i-1} < \delta$ we have that $|w - z| < \delta$. Thus, $|f(w) - f(z)| < \frac{\epsilon}{(b-a)}$ because of our choice of delta. Therefore, if you take $M_i - m_i < f(w) + \theta - (f(z) - \theta) = f(w) - f(z) + 2\theta < \frac{\epsilon}{(b-a)} + 2\theta$. [and θ is arbitrary]. Thus, $M_i - m_i \leq \frac{\epsilon}{(b-a)}$. Thus, $U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leq \frac{\epsilon}{(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\epsilon}{(b-a)}(b-a) = \frac{\epsilon}{2} < \epsilon$. Thus by Riemann's criterion, $f \in \mathcal{R}[a,b]$.

18.2 Derivatives and Integrals

Let $f : [a,b] \to \mathbb{R}$ be a function. We say f is differentiable at a point in $x \in [a,b]$ if there is a number $c \in \mathbb{R}$ such that $\forall \epsilon > 0$, $\exists \delta > 0$ for which we have $\forall y \in [a,b]$, $|y-x| < \delta$, $y \neq x \Rightarrow |\frac{(f(x)-f(y)}{x-y} - c| < \epsilon$.
In other words, $\lim\limits_{y \to x} \frac{f(x)-f(y)}{x-y} = c$. In this case, we denote c by $f'(x)$. This c is unique.

Fact: If f is differentiable at x, then it is continuous at x.
Proof: Take $\epsilon = 1$, then $\exists\ \delta > 0$ such that $|x-y| < \delta$, $y \in [a,b]$, $y \neq x \Rightarrow$ $\frac{f(x)-f(y)}{x-y} - f'(x)| < 1$. This can be written as $|f(x)-f(y)-f'(x)(x-y)| < |x-y|$ which implies by the triangle inequality $|f(x)-f(y)| \leq (|f'(x)|+1)(x-y)$. Take any $\theta > 0$. Then for any y such that $|x-y| < \min\{\frac{\theta}{|f'(x)|+1}, d\}$, we have $|f(x)-f(y)| < (|f'(x)|+1)|x-y| < \theta$.

18.3 Fundamental Theorem of Calculus

Theorem 18.2 (Fundamental Theorem of Calculus) *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function that is differentiable at all $x \in [a,b]$. Suppose that $f' \in \mathcal{R}[a,b]$. Then,*
$\int_a^b f'(x)dx = f(b) - f(a)$

Proof:

Theorem 18.3 *Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$. Then $\exists$ some $c \in [a,b]$ where the derivative $f'(c)$ is 0.*

Proof: Suppose that f is constant in the interval [a,b]. Then there is nothing to prove since $f'(x) = 0\ \forall x \in [a,b]$. Then there are two other cases we have to consider.

If f is not constant in [a,b], there are two possibilities.

- $\exists x \in [a,b]$ where $f(x) > f(a) = f(b)$.
- $\exists x \in [a,b]$ where $f(x) < f(a) = f(b)$.

Both case 1 and case 2 may simultaneously hold, but it suffices to show both independently.

We will only consider Case 1, since Case 2 is similar.
f is differentiable $\Rightarrow$ f is continuous $\Rightarrow$ f attains its maximum value in $[a,b]$ (a is closed and bounded, so compact).

Proof: $[a,b]$ compact $\Rightarrow$ f is bounded $\Rightarrow \sup_{x\in[a,b]} f(x) \in \mathbb{R}$, $\exists\{x_n\}_{n=1}^{\infty} \in [a,b]$ so that $f(x_n) \to \sup_{x\in[a,b]} f(x)$. Find a convergent subsequence x_{n_k} converging to some $x^* \in [a,b]$ [because bounded] which implies $f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = \sup_{x\in[a,b]} f(x)$ [by continuity of f and by any subsequence converges to the same limit as the sequence itself].

Let x^* be such a point (where f is maximized). Since $f(x) > f(a) = f(b)$ for some $x \in [a,b]$, the supremum of $f(x)$: $\sup_{x\in[a,b]} f(x) > f(a) = f(b)$. Thus, $x^* \in (a,b)$ [not including the endpoints].

Since $f(x^*) \geq f(y) \forall y \in [a,b]$, we have that $\frac{f(x^*)-f(y)}{x^*-y} \forall y \in [a,b], y < x^*$. and $\frac{f(x^*)-f(y)}{x^*-y} \leq 0 \forall y \in [a,b], y > x^*$ [since denominator flips signs].

Take any $\epsilon > 0$. Find δ such that $|\frac{f(x^*)-f(y)}{x^*-y} - f'(x^*)| < \epsilon$ whenever $|x^* - y| < \delta$, $y \neq x^x$, $y \in [a,b]$. Find y such that $y > x^*$, $|y - x^*| < \delta$, $y \in [a,b]$ ($\exists$ such y since $x^* \in (a,b)$). Then $f'(x^*) = f'(x^*) - \frac{f(x^*)-f(y)}{x^*-y} + \frac{f(x^*)-f(y)}{x^*-y}$ and $f'(x^*) - \frac{f(x^*)-f(y)}{x^*-y} < \epsilon$ and $\frac{f(x^*)-f(y)}{x^*-y} \leq 0$. Therefore, together $< \epsilon$.

This holds for all $\epsilon > 0$. So $f'(x^*) \leq 0$. Consider $y < x^*$, we similarly get $f'(x^*) \geq 0$.

19 November 14

19.1 Rolle's and Mean Value Theorem

Theorem 19.1 (Rolle's Theorem) *If $f : [a,b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$, then $\exists c \in (a,b)$ where $f'(c) = 0$.*

Used: continuous function on a compact interval attains a maximum.

Generalization of this result:

Theorem 19.2 (Mean Value Theorem) *If $f : [a,b] \to \mathbb{R}$ is differentiable then $\exists c \in (a,b)$ such that $f(a) - f(b) = f'(c)(b-a)$.*

Proof:
Define $g(x) = (f(b)-f(a))x + (b-a)f(x)$. Then g is also differentiable (because x is differentiable, f(x) is differentiable, and sum of two differentiable functions is differentiable). $g(a) = (f(b) - f(a))a - (b-a)f(a) = f(b)a - bf(a)$.

$g(b) = (f(b) - f(a))b - (b-a)f(b) = f(b)a - bf(a) = g(a)$. So g is equal at b and a, thus, $\exists c \in [a,b]$ where $g'(c) = 0$, but $g'(c) = f(b) - f(a) - (b-a)f'(c)$ which gives us the mean value theorem.

Theorem 19.3 (Fundamental Theorem of Calculus) *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f' \in \mathcal{R}[a,b]$. Then $f(b) - f(a) = \int_a^b f'(x)dx$*

Very few conditions on f' – it doesn't have to be continuous, simply f' being Riemann integrable is sufficient. *Proof:* Take any partition $P = \{x_0, ..., x_n\}$ of $[a, b]$. Then $f(b) - f(a) = \sum_{i=1}^{n}(f(x_i) - f(x_{i-1})$ [for any partition, you can write f(b) - f(a) as this sum]. By the mean value theorem, $(f(x_i) - f(x_{i-1})$ can be written as $f'(y_i)(x_i - x_{i-1})$ for some $y_i \in (x_{i-1}, x_i)$. So we have $\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}) \leq \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ $M_i = \sup_{x\in[x_{i-1},x_i]} f'(x)$ and $\geq \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ and $m_i = \inf_{x\in[x_{i-1},x_i]} f'(x)$. So we have $L(f', P) \leq f(b) - f(a) \leq U(f', P)$. Thus $\sup_P L(f', P) \leq f(b) - f(a) \leq \inf_P U(f', P)$. Where $\sup_P L(f', P)$ is the lower Riemann integral: $\int_{\overline{a}}^{b} f'(x)dx$ and $\inf_P U(f', P) = \int_{a}^{\overline{b}} f'(x)dx$. Thus if $f' \in \mathcal{R}[a, b]$, then $f(b) - f(a) = \int_a^b f'(x)dx$.

19.2 Countable and Uncountable Sets

Def. A bijection between two sets X and Y is a map $f : X \to Y$ which is 1-1 and onto. [One-to-one means $f(x) = f(x') \Rightarrow x = x'$ and onto means that $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$].

When there is a bijection between X and Y, we say that these sets have the same cardinality [number of unique elements].

Def. We say that a set X is countable if it has the same cardinality as $\mathbb{N} = \{1, 2, ...\}$. That is, $\exists$ a bijection $f : \mathbb{N} \to X$. If we write $x_i = f(i)$ $i \in \mathbb{N}$, then $x_1, x_2, x_3, ...$ is called an enumeration of X [this is sufficient for proving a bijection: define $x_i = f(x_i)$ for all i].

Theorem 19.4 *Any subset of* $\mathbb{N}$ *is countable or finite.*

Note: Countable often means countably infinite, but can also be used to refer to sets that are either countably infinite or finite.. *Proof:* Suppose $A \subseteq \mathbb{N}$ is infinite. Let a_1 be the minimum element of A (any set of $\mathbb{N}$ has a minimum element – this is an axiom of $\mathbb{N}$). Let a_2 be the minimum of $A\backslash\{a_1\}$ and let a_3 be the minimum element of $A\backslash\{a_1, a_2\}$. Then $a_1, a_2, a_3, ...$ is an enumeration of A. Therefore it is countable.
Fact: If X and Y have the same cardinality, and Y and Z have the same cardinality, then X and Z have the same cardinality.

Corollary 19.4.1 *Any subset of a countable set is finite or countable.*

Proof: Can take the corresponding subset of natural numbers which aligns with the subset you chose to achieve a bijection.

Fact: NxN is countable. Can also do $f(m, n) = 2^m 3^n$ is a 1-1 map from $\mathbb{N}x\mathbb{N}$ into a subset of $\mathbb{N}$. Can tell that $2^m 3^n$ maps to a unique value in $\mathbb{N}$ by prime factorization. In general, any finite number of copies is countable.

Thus any set X that can be written as $\{x_{ij}\}_{i=1,j=1}^{\infty,\infty}$ is countable.

Theorem 19.5 *Any arbitrary countable union of countable set is countable.*

Proof: See course textbook.

Theorem 19.6 (Cantor's diagonalization argument) *[0,1) is not countable.*

Suppose not. Let $x_1, x_2, x_3, ...$ be an enumeration of [0,1]. Let $0 * a_{i,1}a_{i,2}a_{i,3}, ...$ be the decimal expansion of x_i.
Choose $b_1 \in \{0, 1, ..., 9\}$ such that $b_1 \neq a_{1,1}$ [first decimal digit of a_1], $b_2 \in \{0, 1, ..., 9\}$ such that $b_2 \neq a_{2,2}$. $b_3 \neq a_{3,3}$. Let $x = 0 * b_1b_2b_3....$ Then $x \neq x_i$ $\forall i$ since x and x_i do not match in the i^{th} digit. Therefore, the interval $[0, 1]$ is not countable – cannot count the numbers.

Theorem 19.7 $\mathbb{Q}$ *is the set of rational numbers is countable.*

Proof:
$\mathbb{Q} = \{0\} \cup \{\frac{\pm p}{q} | p, q \in \mathbb{N}, q \neq 0, p, q$ are coprime $\}$. Therefore, the set of rational numbers is countable.

Def. A set $X \subseteq \mathbb{R}$ is said to have "measure zero" if $\forall \epsilon > 0$, $\exists$ open intervals $I_1, I_2, ...$ such that $X \subseteq \cup_{i=1}^{\infty} I_i$ and $\sum\limits_{i=1}^{\infty} |I_i| < \epsilon$. $[I = (a, b)$, $|I| = (b - a)]$. (I_i is allowed to be empty).
Effectively means that X is "very small".

19.3 Lebesgue Characterization Theorem

Theorem 19.8 (Lebesque's characterization Theorem) *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuity points of f has measure zero. That is, f is continuous "almost everywhere".*

Something holds "almost everywhere" if the exception has measure zero.
A single point has measure zero. Any finite number of points has measure zero.
Lebesque's great revelation was this characterization of measure theory: i.e. the definition of only countable unions (finite number of unions does not work), but this was a surprise because Riemann has nothing "countable" in it.

20 November 16

20.1 Measure Zero

A set $X \subseteq \mathbb{R}$ is said to have measure 0 if $\forall \epsilon > 0$, $\exists$ countable collection of open intervals $I_1, I_2, ...$ such that $X \subseteq \cup_{i=1}^{\infty} I_i$ and $\sum\limits_{i=1}^{\infty} |I_i| < \epsilon$. Where $I = (a, b)$ and $|I| = b - a$.
Ex. Any finite set has measure zero. Let $X = \{x_1, .., x_n\}$ and $I_i = (x_i - \frac{\epsilon}{4n}, x_i + \frac{\epsilon}{4n})$ and $|I_i| = \frac{\epsilon}{2n}$ which when you sum up all the I_is you get less than ϵ, hence this set has measure 0.

Theorem 20.1 *If $X_1, X_2, ...$ has measure zero, then so does $\cup_{i=1}^{\infty} X_i$ [countable union of countable sets].*

Any countable set has measure zero since it's the countable set of measurable points. Major difference between measure theory and topology: topology can be arbitrary (arbitrary intersection of closed sets is closed), but in measure theory, you need a countable number of unions.
Proof:
Fix $\epsilon > 0$. For each i, let $I_{i,1}, I_{i,2}, ...$ be a sequence of open sets such that $X_i \subseteq \cup_{j=1}^{\infty} I_{i,j}$ and the sum of the lengths $\sum_{j=1}^{\infty} |I_{i,j}| < \frac{\epsilon}{2^i}$.

Consider the collection $\{I_{i,j}\}_{i \geq 1, j \geq 1}$. Then $\cup_{i=1}^{\infty} X_i \subseteq \cup_{i=1}^{\infty} \cup_{j=1}^{\infty} I_{i,j}$. We can enumerate this collection $\cup\cup I_{i,j}$ as $J_1, J_2, ...$ (this is simply $\mathbb{N} \times \mathbb{N}$), and $\cup_{i=1}^{\infty} X_i \subseteq \cup_{j=1}^{\infty} J_j$. Now for any n, $\sum_{j=1}^{n} |J_j| \leq \sum_{i=1}^{n'} \sum_{j=1}^{m'} |I_{i,j}|$ for some large enough n', m'. And this is $\leq \sum_{i=1}^{n'} (\sum_{j=1}^{\infty} |I_{i,j}|)$ where $\sum_{j=1}^{\infty} |I_{i,j}| < \frac{\epsilon}{2^i}$ and $\sum_{i=1}^{n'} \frac{\epsilon}{2^i} \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$. Thus $\forall n$, $\sum_{j=1}^{n} |J_j| < \epsilon \Rightarrow \sum_{j=1}^{\infty} |J_j| \leq \epsilon$.

20.2 Lebesgue Characterization Theorem: I

Theorem 20.2 (Lebesgue Theorem) *Let $f : [a, b]$ be a founded function. Then $f \in \mathcal{R}[a, b]$ if and only if the set of discontinuity points of f has measure zero.*

Proof:
Need to start with several lemmas.

For any set $X \subseteq \mathbb{R}$, the "oscillation" of f in X is defined as $\omega_f(X) = \sup_{x \in X \cap [a,b]} f(x) - \inf_{x \in X \cap [a,b]} f(x)$. X can be any set, but we only take the supremum and infimum over the closed intersection $[a, b]$. f is defined on this interval. Leave it undefined if X does not intersect $[a, b]$. Finite quantity since f is a bounded function.

The oscillation of f at a point $x \in [a, b]$ is defined as:

$$\omega_f(x) = \inf\{\omega_f(I) | I \text{ is an open interval containing x}\}$$

Lemma 20.3 *f is continuous at x if and only if $\omega_f(x) = 0$.*

Proof:
Suppose that f is continuous at x. Take any $\epsilon > 0$. Then $\exists\delta > 0$ such that if $y \in [a,b]$, $|y-x| < \delta$, then $|f(x)-f(y)| < \epsilon$ [by continuity]. Thus if $I = (x-\delta, x+\delta)$, then $\forall y \in I \cap [a,b]$, $f(y) \in (f(x)-\epsilon, f(x)+\epsilon) \Rightarrow \sup_{y\in I\cap[a,b]} f(y) \leq f(x)+\epsilon$ and $\inf ... \geq f(x)-\epsilon$. Therefore $\omega_f(I) \leq 2\epsilon$. But $w_f(x) \leq w_f(I) \leq 2\epsilon \ \forall\epsilon > 0$ [a number is less than or equal to ϵ for all $\epsilon > 0$, therefore mus be negative or 0, and the oscillation is non-negative, hence $w_f(x) = 0$]. Thus, $w_f(x) = 0$.

Conversely, suppose that $w_f(x) = 0$. Take any $\epsilon > 0$. Then, since the oscillation $w_f(x) = \inf\{w_f(I)|I$ is an open interval containing x$\}$, we have that $\exists I$ such that $x \in I$, I is an open interval, such that $w_f(I) < \epsilon$ [any number bigger than greatest lower bound cannot be a lower bound].

Since I is an open set, $\exists\delta > 0$ such that $(x-\delta, x+\delta) \subseteq I$. $\Rightarrow w_f((x-\delta, x+\delta)) \leq w_f(I) < \epsilon$. (Note if $X \subseteq Y$, then $w_f(X) \leq w_f(Y)$). Then for any $y \in [a,b]$, where $|y-x| < \delta$, we have $y \in (x-\delta, x+\delta)$ which implies $|f(y)-f(x)| \leq \sup_{z\in(x-\delta,x+\delta)\cap[a,b]} f(z) - \inf_{z\in(x-\delta,x+\delta)\cap[a,b]} f(z) = w_f((x-\delta, x+\delta)) < \epsilon$. Similarly, $f(x) - f(y) < \epsilon$ [by the same argument as above]. These last two statements together imply $|f(x)-f(y)| < \epsilon$ [$f(y)-f(x) < \epsilon$ and $f(x)-f(y) < \epsilon$].

Lemma 20.4 *For any $\epsilon > 0$, the set $\{x \in [a,b] : w_f(x) < \epsilon\}$ is open in [a,b] (means its an intersection of [a,b] with an open intersection of the real line).*

Proof:
Let $A = \{x \in [a,b] : w_f(x) < \epsilon\}$. If $x \in A$, then $w_f(x) < \epsilon$. $\Rightarrow$
$\inf\{w_f(I)|I$ is an open interval containing x.$\} < \epsilon \Rightarrow \exists$ an open interval I with $x \in I$ such that $w_f(I) < \epsilon$ [infimum strictly less than ϵ is critically important]. If we show that $I\cap[a,b] \subseteq A$, this will prove that A is open (since $I\cap[a,b]$ is an open interval in $[a,b]$). But this is true since $\forall y \in I\cap[a,b]$, $\omega_f(y) \leq \omega_f(I) < \epsilon$ $\Rightarrow y \in A$. Need to show that for any point in the set, there's an open interval containing that point in your set. This whole interval I is contained within A, therefore, A is open.

Lemma 20.5 *x is a continuity point of f if and only if*

21 November 18

21.1 Lebesgue Characterization Theorem: II

$f : [a,b] \Rightarrow \mathbb{R}$ bounded function.

Theorem 21.1 (Lebesgue Theorem) *$f \in \mathcal{R}[a,b]$ if and only if the set of discontinuous points of f has measure zero (f is continuous almost everywhere).*

Lemma 21.2 (Lemma 1) *f is continuous at $x \iff w_f(x) = 0$*

Lemma 21.3 (Lemma 2) *$\{x \in [a, b] | w_f(x) < \epsilon\}$ is open in $[a, b]$ for any $\epsilon > 0$.*

Lemma 21.4 (Lemma 3) *If $w_f(x) < \epsilon \ \forall x \in [a, b]$, then $\exists$ partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon(b - a)$.*

[If the oscillations are small on the full interval, bounded by some ϵ for all points in that interval, then there's a partition into finite parts so that the difference between the upper and lower sum is bounded by a value.]
Proof:
Take any $x \in [a, b]$. $\omega_f(x) < \epsilon \Rightarrow \exists$ open interval $x \in J_x$ such that $w_f(J_x) < \epsilon$ [open interval containing x that has oscillation less than ϵ]. Then, $\exists$ open interval $I_x \supseteq J_x$ such that $x \in I_x$ and $\overline{I}_x \supseteq J_x$. Thus, $w_f(\overline{I_x} \leq w_f(J_x) < \epsilon$. Each x has an open interval whose oscillation is less than ϵ.

Note that $\{I_x\}_{x\in[a,b]}$ is an open cover of $[a, b] \Rightarrow \exists\ x_1, ..., x_n \in [a, b]$ such that $[a, b] \supseteq \cup_{i=1}^{n} I_{x_i}$ [closed interval $[a, b]$ is compact $\Rightarrow$ finite subcover].

Let P = set of endpoints of $I_{x_1}, ..., I_{x_n}$ which are in $[a, b]$ together with a and b. All of these things together form a finite set and this gives us our partition. If any endpoints are outside of $[a, b]$, we can simply discard them.

Arrange the elements of P as $a = y_0 < y_1 < ... < y_m = b$. Take any $1 \leq i \leq m$.
Claim: $[y_{i-1}, y_i] \subseteq \overline{I}_{x_j}$ for some j.
Proof:
Take $\frac{y_{i-1}+y_i}{2} \in \cup_{j=1}^{n} I_{x_j} \Rightarrow \frac{y_{i-1}+y_i}{2} \in I_{x_j}$ for some j. This implies that the left endpoint of $I_{x_j} \leq y_{i-1}$ and the right endpoint of $I_{x_j} \geq y_i \Rightarrow [y_{i-1}, y_i] \subseteq \overline{I}_{x_j}$. $\Rightarrow w_f([y_{i-1}, y_i]) \leq w_f(\overline{I}_{x_j})$. We have been able to partition $[a, b]$ so that the oscillation of each block of the partition is less than ϵ, and the finite number of partitions comes from compactness of $[a, b]$.

$U(f, P) - L(f, P) = \sum_{i=1}^{n} \sup_{x\in[y_{i-1},y_i]} f(x)(y_i - y_{i-1}) - \sum_{i=1}^{n} \inf x \in [y_{i-1}, y_i] f(x)(y_i - y_{i-1}) = \sum_{i=1}^{n} w_f([y_{i-1}, y_i])(y_i - y_{i-1}) < \epsilon \sum_{i=1}^{n} (y_i - y_{i-1}) = \epsilon(b - a)$

Proof of Lebesgue's Theorem First, suppose that $f \in \mathcal{R}[a, b]$. Let X = set of discontinuity points of f.
To Show: X has measure zero.
By Lemma 1, $X = \{x \in [a, b] | \omega_f(x) > 0\} = \cup_{n=1}^{\infty} \{x \in [a, b] | \omega_f(x) \geq \frac{1}{m}\}$.

Since a countable union of sets of measure zero has measure zero, it suffices to prove that each X_m has measure zero. If you can prove for any m, $\cup_{n=1}^{\infty} \{x \in [a, b] | \omega_f(x) \geq \frac{1}{m}\}$ has measure zero, then we're done. Take any m. We now show that X_m has measure 0. Take any $\epsilon > 0$, we have to find a countable cover of open sets that has ... Since $f \in \mathcal{R}[a, b]$, there is a partition

P of $[a,b]$ such that $U(f,P) - L(f,P) < \frac{\epsilon}{2m}$ [by Riemann's criteria, there is such a partition]. Write $X_m = A \cup B$ where $A = X_m \cap P$ and $B = X_m \backslash P$. X_m is potentially an infinite set, it may also be an empty set, but we divide it into two parts and treat them separately. $P = \{y_0, ..., y_n\}$. If $x \in B$, then $x \in (y_{i-1}, y_i)$ for some i. Let $i_1, ..., i_k$ be the indices i such that (y_{i-1}, y_i) intersects X_m. Then $\cup_{j=1}^k (y_{i_{j-1}}, y_{i_j}) \supseteq B$. But if $x \in B$ and $x \in (y_{i-1}, y_i)$, then $w_f([y_{i-1}, y_i]) \geq w_f((y_{i-1}, y_i)) \geq w_f(x) \geq \frac{1}{m}$ since $x \in X_m$. $\Rightarrow \sum_{j=1}^k w_f([y_{i_{j-1}} - y_{i_j}])(y_{i_j} - y_{i_{j-1}}) \geq \sum_{j=1}^k \frac{1}{m}(y_{i_j} - y_{i_{j-1}})$. If you just take those intervals that intersect with X_m in their interior, this inequality holds. Moreover, $\sum_{j=1}^k w_f([y_{i_{j-1}} - y_{i_j}])(y_{i_j} - y_{i_{j-1}}) \leq \sum_{i=1}^n w_f([y_{i-1}, y_i])(y_i - y_{i-1}) = U(f,P) - L(f,P) < \frac{\epsilon}{2m}$. Thus, $\sum_{j=1}^k (y_{i_j} - y_{i_{j-1}}) < \frac{\epsilon}{2}$. But $\{(y_{i_j} - y_{i_{j-1}})\}_{j=1}^k$ is a collection of open intervals that covers B. Now A is a finite set, and so it has measure zero. Recall $A = X_m \cap P$ and $B = X_m \backslash P$. So we can find a countable (actually finite since A is a finite set) cover of A by open intervals with total length $< \frac{\epsilon}{2}$. Putting this together with $\{(y_{i_{j-1}}, y_{i_j})\}_{i=1}^k$ we get a cover of X_m by finitely many open intervals with total length $< \epsilon$. Since ϵ is arbitrary, X_m has measure zero. Therefore, we've proved one way...

Used countability to say that if each X_m has measure zero, then the countable union of them has measure zero.

Question about oscillations: what is the oscillation at a point? $w_f(x) = \inf\{w_f(I) | I \text{ open interval containing x}\}$.
Converse is also true..

22 November 28

22.1 Lebesgue Characterization Theorem: III

Remaining part of the proof of the Lebesgue's characterization theorem.
Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function which is continuous almost everywhere (i.e. the set of discontinuity points has measure zero).
To Show: f is Riemann Integrable.
Proof: Let $X_m = \{x \in [a,b] | w_f(x) \geq \frac{1}{m}\}$. Then the set of discontinuity points $= \{x | w_f(x) > 0\} = X = \cup_{m=1}^{\infty} X_m$.

We know that X has measure zero. For each m, $X_m \subseteq X$. So X_m also has measure zero (since any covering of X by open sets is also a covering of X_m). Fix any $\epsilon > 0$. We will now find a partition P such that $U(f,P) - L(f,P) < \epsilon$. By Riemann's Criterion, this shows that f is Reimann integrable.

Find m such that $\frac{b-a}{m} < \frac{\epsilon}{2}$. If $w_f([a,b]) = 0$, then f is constant in $[a,b]$ and therefore Reimann Integrable (R.I.). So, let's assume that $w_f([a,b]) > 0$. Since X_m has measure zero, $\exists$ open intervals $I_1, I_2, ...$ such that $X_m \subseteq \cup_{i=1}^{\infty} I_i$ and

$\sum\limits_{i=1}^{\infty}|I_i| < \frac{\epsilon}{2w_f([a,b])}$.

Recall that by Lemma 2, X_m is closed in $[a,b]$. But $[a,b]$ is compact. So X_m is compact (any closed subset of a compact set is compact). Thus, $\{I_i\}_{i=1}^{\infty}$ has a finite subcover of X_m. So $\exists n$ such that $X_m \subseteq \cup_{i=1}^{n} I_i$. Note that $\sum\limits_{i=1}^{n}|I_i| < \sum\limits_{i=1}^{\infty}|I_i| < \frac{\epsilon}{2w_f([a,b])}$. Now, note that $\cup_{i=1}^{n} I_i$ is a union of disjoint open intervals$J_1, ..., J_k$ (prove this by induction).

Moreover, $\sum_{i=1}^{k}|J_i| \leq \sum\limits_{i=1}^{n}|I_i|$ (again, prove by induction on n). Take the endpoints of these intervals that are in $[a,b]$ together with a,b (i.e. take the endpoints of the open intervals along with the points a,b). This gives a partition $Q = \{y_0, y_1, ..., y_l\}$ of $[a,b]$.

Take any $1 \leq i \leq l$. Then either (y_{i-1}, y_i) is contained in of the $J_j's$ (Type 1: i.e. it is strictly inside [a,b]) or $[y_{i-1}, y_i] \subseteq [a,b] \backslash X_m$ (Type 2: i.e. the whole interval does not intersect X_m – is outside the interval).

Recall: $U(f,Q) - L(f,Q) = \sum_{i=1}^{l} w_f([y_{i-1}, y_i])(y_i - y_{i-1})$ [oscillation of f in this interval multiplied by the length of the interval].
$= \sum_{i \in Type1} + \sum_{i \in Type2}$ – can break up the sum into these two parts.

If $i \in$ Type 1, then $(y_i - y_{i-1}) \leq$ the length of the J_j containing this interval, and $w_f([y_{i-1}, y_i]) \leq w_f([a,b])$. Thus, $\sum_{i \in type1} \leq w_f([a,b]) \sum_{j=1}^{k}|J_j| < w_f([a,b])\frac{\epsilon}{2w_f([a,b])} = \frac{\epsilon}{2}$.

If $i \in$ Type 2, then $w_f(x) < \frac{1}{m}\ \forall x \in [y_{i-1}, y_i]$. By Lemma 3, we can find a partition P_i of $[y_{i-1}, y_i]$ such that $U(f,P) - L(f,P) < \frac{y_i - y_{i-1}}{m}$. Refine Q by subdividing $[y_{i-1}, y_i]$ into P_i. Do this $\forall$ Type 2 i. Let P be this refinement of Q (take the type 1 intervals – leave them as they are – and take the type 2 intervals and subdivide them). $U(f,P) - L(f,P) =$ Contributions from type 1 + Contributions from subintervals of type 2. Upper bound on contributions from type 1 is bounded by $\frac{\epsilon}{2}$ (keeping them as they are – not changing them). Contributions from type 2 $\leq \frac{\sum lengths}{m} \leq \frac{b-a}{m} < \frac{\epsilon}{2}$ after doing the partitions is the sum of the lengths divided by m. In Type 2, intervals are all disjoint, total contribution is the sum across all i for $U(f,P_i) - L(f,P_i) < \frac{y_i - y_{i-1}}{m} = \frac{\sum lengths}{m}$.

23 November 30

23.1 Lebesgue Integration

This will be covered properly in Math 172. Riemann Integral has some nice properties:

- $\int_a^b (f+g)dx = \int_a^b f dx + \int_a^b g dx$

One problem with the Riemann integral is that $\{f_n\}_{n\geq 1}$ is a sequence of functions converging pointwise to f, then f may not be Riemann integrable even if each f_n is Riemann Integrable and the entire sequence is uniformly bounded. If not closed under pointwise limits, then there are issues.

Example: Let $q_1, q_2, q_3, ...$ be an enumeration of the rational numbers in $[0,1]$. Let $f_n(x)$ be 0 if $x \in \{q_1, ..., q_n\}$ and 1 otherwise. $f_n : [0,1] \to \mathbb{R}$. $\int_0^1 f_n(x)dx = 1$ since $f_n = 1$ almost everywhere (except on a finite set – measure zero). Each f_n is Riemann Integrable, but $f_n \to f$ pointwise where $f(x) = \begin{cases} 0 \text{ if } x \in \mathbb{Q} \cup [0,1] \\ 1 \text{otherwise} \end{cases}$.

In any interval $[x,y] \subseteq [0,1]$, $x < y$, $\sup_{t\in[x,y]} f(x) = 1$ $\inf_{t\in[x,y]} f(t) = 0$. This implies for any partition P, $U(f,P) = 1$ and $L(f,P) = 0$. Therefore, f is not Riemann integrable.

23.2 Lebesgue Outer Measure

For any $A \subseteq R$, define $\lambda^*(A) = \inf\{\sum_{i=1}^{\infty} |I_i| | I_1, I_2, ...$ are open intervals and $A \subseteq \cup_{i=1}^{\infty} I_i\}$. Can define the length of a set to be this value.

Let's verify this works for intervals.

Lemma 23.1 *If $A = [a,b]$ for some $a < b$, then $\lambda^*(A) = b - a$*

Proof:
Take any $\epsilon > 0$. Then $A \subseteq (a-\epsilon, b+\epsilon)$. Thus $\lambda^*(A) \leq (b+\epsilon)-(a-\epsilon) = b-a+2\epsilon$. Thus, $\lambda^*(A) \leq b - a$.

Take any sequence of open intervals $I_1, I_2, ...$ such that $A \subseteq \cup_{i=1}^{\infty} I_i$, and A is a closed interval, so by compactness of A, $\exists n$ such that $A \subseteq \cup_{i=1}^{n} I_i$. Recall that $\cup_{i=1}^{n} I_i$ can be written as a disjoint union of open intervals $J_1, ..., J_k$ and also that $\sum_{i=1}^{k} |J_i| \leq \sum_{i=1}^{n} |I_i|$.

Now since $J_1, ..., J_k$ are disjoint open intervals, and their union contains $[a,b]$, one of them has to entirely contain $[a,b]$. The length of that J_i must be $\geq b - a$ (must actually be strictly bigger..). Thus $b - a \leq \sum_{i=1}^{k} |J_i| \leq \sum_{i=1}^{n} |I_i| \leq \sum_{i=1}^{\infty} |I_i|$. Thus, $b - a \leq \lambda^*(A)$. This is because our collection I_i is any collection of open intervals whose union contains $[a,b]$. Therefore, $b - a \leq \lambda^*(A)$.

- $\lambda^*(\emptyset) = 0$
- If $A \subseteq B$, then $\lambda^*(A) \leq \lambda^*(B)$

Want: If $A_1, A_2, ...$ are disjoint sets, then $\lambda^*(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda^*(A_i)$ (denote this statement as (*). Unfortunately, this is not true (because of the axiom of choice).

Theorem 23.2 *(*) is not true.*

Proof:
Define a relation $\sim$ on [0,1] as follows: $x \sim y$ if $x - y \in \mathbb{Q}$. Can easily check that this is an equivalence relation, which means that
Meaning of an equivalence relation.

1. $x \sim x$
2. $x \sim y \Rightarrow y \sim x$
3. $x \sim y,\, y \sim z, \Rightarrow x \sim z.$

Therefore, we can break up a set into equivalence classes.

Fact: Any equivalence relation on any set splits the set into disjoint equivalence classes. (An equivalence class is the set of all y such that $y \sim x$ for some given x). Two equivalence classes cannot overlap, and one element can only be in a single equivalence class.

By the axiom of choice, there is a set $A \subseteq [0, 1]$ consisting of exactly one element from each equivalence class. Such sets are called Vitali sets.

The axiom of choice says that given any collection of sets, there is a set consisting of exactly one element from each of these sets that you have. Consistent of other axioms of set theory, but cannot be proven from the other axioms. The problem arises, because not all collections are sets.

Russell's Paradox: There cannot be a set of all sets.
Proof:
Suppose not. Let A be the set of all sets. If we define $B = \{x \in A | x \notin x\}$ is also a set. If $B \in B$, then $B \notin B$. On the other hand, if $B \notin B$, then $B \in B$. So in either case, you arrive at a contradiction.

Claim: If (*) is true (where (*) is the statement that for any disjoint $A_1, A_2, ... \subseteq \mathbb{R}$), $\lambda^*(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda^*(A_i)$), then $\lambda^*(A)$ cannot be zero and cannot be nonzero.
Proof:
For any $r \in \mathbb{Q} \cap [-1, 1]$, let $A(r) = A + r$ (set of $\{a + r | a \in A\}$.

Easy: $\lambda^*(A + r) = \lambda^*(A)$. [translate of A].
Claim: $\{A_r\}_{r \in \mathbb{Q} \cap [-1,1]}$ are disjoint.
Proof:

(next class)

Proof of Claim 1:
Let $B = \cup_{r\in\mathbb{Q}\cap[-1,1]} A_r$ [by claim 2 disjoint]. $\Rightarrow \lambda^*(B) = \sum_r \lambda^*(A_r)$
$= \sum_r \lambda^* A$
$= \begin{cases} \infty \text{ if } \lambda^*(A) > 0 \\ 0 \text{ if } \lambda^*(A) = 0 \end{cases}$
Claim: $[0,1] \subseteq B \subseteq [-1,2] \Rightarrow 1 \leq \lambda^*(B) \leq 3$.

24 December 2

24.1 $\sigma-$algebras

$x \sim y$ if $x - y \in \mathbb{Q}$, $x \in [0,1]$. $A =$ a set which consists of exactly one element from each equivalence class.

$A_r = A + r = \{a + r | a \in A\}$ for $r \in \mathbb{Q} \cup [-1,1]$. So $\lambda^*(A_r) = \lambda^*(A)\ \forall r$.

Claim: $A_r \cap A_s = \emptyset$ *if* $r \neq s$
Proof:
Suppose $x \in A_r \cap A_s$. Then $x \in A_r \Rightarrow \exists b \in A$ such that $x = b + r$.
$x \in A_s \Rightarrow \exists c \in A$ such that $x = c + s$.
Thus, $b + r = c + s \Rightarrow b - c \in \mathbb{Q}\backslash\{0\}$ (since $r, s \in \mathbb{Q}$, $r \neq s$). But then $b, c \in A$, $b \neq c$ and $b \sim c$. This is impossible. [No two elements of A can be equal and equivalent]. Let $B = \cup_{r\in\mathbb{Q}\cap[-1,1]} A_r$. This is a countable disjoint union of subsets of $\mathbb{R}$.

Claim: $\lambda^*(B) \neq \sum_{r\in\mathbb{Q}\cap[-1,1]} \lambda^*(A_r)$
Proof:
Suppose not. Then $\lambda^*(B) = \begin{cases} \infty \text{ if } \lambda^*(A) > 0 \\ 0 \text{ if } \lambda^*(A) = 0 \end{cases}$. We will show that both cases are impossible.

Claim: $[0,1] \subseteq B \subseteq [-1,2]$
$\lambda^*(B)$ would be between 1 and 3 since $\lambda^*([0,1]) = 1$ and $\lambda^*([-1,2]) = 3$.

Take any $x \in [0,1]$. Then $x \sim b$ for some $b \in A$. $\Rightarrow x = b + r$ for some $b \in A$ and $r \in \mathbb{Q}$. But $r = x - b \in [-1,1]$ since $x, b \in [0,1]$. Thus $r \in \mathbb{Q} \cap [-1,1]$ and $x \in A_r$. Therefore, $[0,1] \subseteq B$.

WTS $B \subseteq [-1,2]$. Take any $x \in B$. Then $x \in A_r$ for some $r \in \mathbb{Q} \cap [-1,1] \Rightarrow$ $x = b + r$ for some $b \in A$ and $r \in \mathbb{Q} \cap [-1,1]$. $A \subseteq [0,1]$, so $b \in [-1,2]$.

So we just proved that the length of the union of disjoint subsetes under

Lebesgue's outer measure is not always equal to the sum of the measures of each disjoint subset.

Theorem 24.1 *We cannot define a notion of length that has all the following properties:*

1. $0 \leq$ *length(A)* $\leq \infty$ *for all* $A \subseteq \mathbb{R}$
2. *If* $A \subseteq B$*, then* $length(A) \leq length(B)$
3. *If* B *is a translate of* A*, then length(B) = length(A)*
4. $length([a, b]) = b - a$ *for any* $a < b$
5. *If* $A_1, A_2, \ldots$ *are disjoint, then the length*$(\cup_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} length(A_i)$

No sensible way to define length of a set for arbitrary sets (if we assume the axiom of choice).

Legesgue wanted to define a new integral. Needed the length of an arbitrary subset of the real numbers (i.e. the length where some function takes a specific value). Then multiply this length by the function value (and repeat for all values).

$f(x) = \begin{cases} 1 \text{ if } x \notin \mathbb{Q} \\ 0 if x \in \mathbb{Q} \end{cases}$ Then $f(x)$ takes value 1 in the irrational numbers: this set has measure 1, so 1*1 = 1. $f(x)$ takes value 0 in the rational numbers which has measure 0: 0*0 = 0. So 0 + 1 = 1 which is the Lebesgue integral of this function.

Theorem 24.2 (Sigma Algebra) *A subset* $\mathcal{F}$ *of the power set of* $\mathbb{R}$ *is called a* σ*-algebra (or* $\sigma - field$*) if*

1. $\emptyset \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$
3. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Ex. $\{\emptyset, \mathbb{R}\}$ is a $\sigma - algebra$.
Ex. $P(\mathbb{R})$ the power set of $\mathbb{R}$ is a $\sigma - algebra$.

24.2 Borel $\sigma-$algebras

Borel $\sigma - algebra$
$\mathcal{B}(\mathbb{R})$ is the set of all subsets $A \subseteq \mathbb{R}$ such that $A \in \mathcal{F}$ for every $\sigma - algebra$ $\mathcal{F}$ that contains all the open sets.
$P(\mathbb{R})$ is the power set of $\mathbb{R}$ contains all the open sets. Every open set, singleton, $\emptyset$, $\mathbb{R}$, and closed set is in $\mathcal{B}(\mathbb{R})$.

Lemma 24.3 *$\mathcal{B}(\mathbb{R})$ is a $\sigma - algebra$.*

Proof:
(1) $\phi \in \mathcal{F}\ \forall \mathcal{F} \Rightarrow \phi \in \mathcal{B}(\mathbb{R})$.

(2) If $A \in \mathcal{B}(\mathbb{R})$, then for any $\sigma - algebra \mathcal{F}$ containing all open sets, $A \in \mathcal{F}$. Thus $A^C \in \mathcal{F}$. And so, $A^C \in \mathcal{B}(\mathbb{R})$.

(3) If $A_1, A_2, ... \in \mathcal{B}(\mathbb{R})$, then for any $\sigma - algebra\ \mathcal{F}$ containing all open sets, $A_1, A_2, ... \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{B}(\mathbb{R})$. So $\mathcal{B}(\mathbb{R})$ is a $\sigma - algebra$.

Fact: If $\mathcal{F}$ is a $\sigma - algebra$ and $A_1, A_2, ... \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.
Proof: $\cap A_i = ((\cap A_i)^C)^C = (\cup A_i^C)^C$.

$[a, b] = \cap_{n=1}^{\infty} = \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$, so all closed sets are $\in \mathcal{B}(\mathbb{R})$.

Theorem 24.4 *If $A_1, A_2, ... \in \mathcal{B}(\mathbb{R})$ are disjoint, then $\lambda^*(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda^*(A_i)$. If $A \in \mathcal{B}(\mathbb{R})$ we denote $\lambda^*(A)$ by $\lambda(A)$ and this is called the Lebesgue measure of A.*

24.3 Measurable functions

Def A function $f : \mathbb{R} \to \mathbb{R}$ is called Borel-measurable (or simply, "measurable") if for any $B \in \mathcal{B}(\mathbb{R})$, the set $f^{-1}(B) = \{x \in \mathbb{R} | f(x) \in B\}$ is also in $\mathcal{B}(\mathbb{R})$.

Inverse image of any Borel set is Borel. This encompasses any function you'll see. The most complicated neural net is also a measurable function.

The indicator of the set A we constructed is not measurable (1 if in A, 0 otherwise).

Lemma 24.5 *Any continuous function is measurable.*

Lemma 24.6 *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions converging pointwise to a limit f, then f is measurable.*

Anything that is a limit of continuous functions is measurable (by pointwise convergence) is measurable. This is an **incredibly strong** statement.

25 December 5

25.1 Measurable Functions

$\mathcal{B}$ = Borel sigma algebra on $\mathbb{R}$ = smallest $\sigma-$algebra containing all open sets = intersection of all σ-algebra on $\mathbb{R}$ that contain all open sets = $\sigma-$algebra generated by the collection of open sets.

$\mathcal{B} = \sigma-$alg. generated by all intervals of the form $(x, \infty) = \sigma-$alg. generated by intervals of the form $(-\infty, x)$.
Proof:
Def. Let $\mathcal{A}$ be a collection of subsets of $\mathbb{R}$. The $\sigma-$algebra generated by $\mathcal{A}$ is the set of all subsets of $\mathbb{R}$ that belong to every $\sigma-$algebra containing $\mathcal{A}$ as a subset.

Let $\mathcal{F}$ be the $\sigma-$algebra generated by all intervals like (x, ∞).

Proof of $\mathcal{F} = \mathcal{B}(\mathcal{R})$:
$\mathcal{B}(\mathbb{R})$ contains $\infty \forall x$ since $\mathcal{B}(\mathbb{R})$ contains all open sets. So $\mathcal{B}(\mathbb{R})$ is a $\sigma-$algebra containing (x, ∞) $\forall x$. Therefore, any $A \in \mathcal{F}$ must be an element of $\mathcal{B}(\mathbb{R})$. Therefore $\mathcal{F} \subseteq \mathcal{B}(\mathbb{R})$.

Take any $A \in \mathcal{B}(\mathbb{R})$. To show: $A \in \mathcal{F}$.
Know that $(x, \infty) \in \mathcal{F}$ $\forall x$ and $\mathcal{F}$ is a $\sigma-$algebra. $\Rightarrow \forall x < y$, $(x, \infty) \backslash (y, \infty) \in \mathcal{F}$: the difference of two sets is in $\mathcal{F} = (x, y]$. $\sigma-$algebras closed under addition, unions, subtractions, intersections, complements, countable unions, countable intersections are in the $\sigma-$algebra. $\Rightarrow (x, y) = \cup_{n=1}^{\infty}(x, y - \frac{1}{n}) \in \mathcal{F}$.

Fact: Any open subset of $\mathbb{R}$ is a countable union of open intervals. (exercise – use the rationals).
$\Rightarrow$ any open set $\in \mathcal{F} \Rightarrow \mathcal{B}(\mathbb{R}) \in \mathcal{F}$.

Recall: A function f: $\mathbb{R} \to \mathbb{R}$ is called (Borel) measureable if $\forall B\mathcal{B} \in \mathcal{B}(\mathbb{R})$, $f^{-1}(\mathcal{B}) \in \mathcal{B}(\mathbb{R})$

Proposition 25.0.1 *Any continuous function is measurable.*

Proof:
Let $\mathcal{F} = \{B \subseteq \mathbb{R}) | f^{-1}(B) \in \mathcal{B}(\mathbb{R})\}$.
Claim: $\mathcal{F}$ is a $\sigma-$algebra.
Proof:
$f^{-1}(\emptyset) = \emptyset \in \mathcal{B}(\mathbb{R})$, and so $\emptyset \in \mathcal{F}$.

Suppose $B \in \mathcal{F}$. Then $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$. Then $f^{-1}(B^C) = \{x : f(x) \in B^C\} = \{x | f(x) \notin B\} = (f^{-1}(B))^C \in \mathcal{B}(\mathbb{R}) \Rightarrow B^C \in \mathcal{F}$ [inverse image of complement of B is in $\mathcal{B}(\mathbb{R})$].

Suppose $B_1, B_2, ... \in \mathcal{F}$. Then $f^{-1}(B_1), f^{-1}(B_2), ... \in \mathcal{B}(\mathbb{R})$. So $f^{-1}(\cup_{i=1}^{\infty} B_i) = \cup_{i=1}^{\infty} f^{-1}(B_i) \in \mathcal{B}(\mathbb{R}) \Rightarrow \cup B_i \in \mathcal{F}$.

Since f is continuous, $f^{-1}(B)$ is open for all open B. Therefore, $f^{-1}(B)$ is a Borel set for all open B. Therefore, any open set belongs to $\mathcal{F} \Rightarrow \mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}$.

This proves our claim.

This shows the class of measurable functions contains at least all continuous functions. Our next goal is that all point-wise limits of measurable functions is itself measurable.

Lemma 25.1 *Let $f_1, f_2, ...$ be measurable functions. Let $g(x) = \inf_{n \geq 1} f_n(x)$ $\forall x \in \mathbb{R}$. Suppose that $g(x) \in \mathbb{R}$ $\forall x$ [infimum is finite]. Then g is measurable.*

Infimum of a collection of measurable functions is measurable.
Proof:
Take any $t \in \mathbb{R}$. Then $g(x) < t$ if and only if $f_n(x) < t$ for some n. Thus, $\{x|g(x) < t\} = \cup_{n=1}^{\infty} \{x|f_n(x) < t\}$. $\{x|g(x) < t\} = g^{-1}((-\infty, t)) \in \mathcal{B}(\mathbb{R})$ and $\{x|f_n(x) < t\}$. $\{x|g(x) < t\} = f_n^{-1}((-\infty, t)) \in \mathcal{B}(\mathbb{R})$.

Since the set (∞, t), $t \in \mathbb{R}$ generate $\mathcal{B}(\mathbb{R})$, this shows (by a similar argument as in the previous proof) that g is measurable.

Up and including to Lebesque Characterization Theorem is on the exam.

Lemma 25.2 *If f_n is measurable for all n and $g(x) = \limsup_{n \to \infty} f_n(x)$ is finite (i.e. is in $\mathbb{R}$) $\forall x$, then g is measurable.*

Proof:
$g(x) = \inf_{k \geq 1} h_k(x)$ where $h_k(x) = \sup_{n \geq k} f_n(x)$. By a counterpart of the previous lemma, each h_k is measurable. Thus, g is measurable (pointwise $\limsup$ gives us a measurable function).

Theorem 25.3 *If $\{f_n\}$ is a sequence of measurable functions converging pointwise to a function f, then f is measurable.*

Proof:
$f(x) = \lim_{n \to \infty} f_n(x) = \limsup_{n \to \infty} f_n(x)$

Therefore, the pointwise sequence of any measurable functions is measurable.

26 December 7

26.1 Lebesgue Integration of Measurable Functions

$\mathcal{B}(\mathbb{R})$: Borel $\sigma-$algebra, $f : \mathbb{R} \to \mathbb{R}$ (measurable function), λ = Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

How to define $\int_{-\infty}^{\infty} f(x)dx$ for all measurable functions.

Def. A measurable function f is called "simple" if it takes only finitely many values.
Ex. The step function on limited domain.
Ex. $\begin{cases} 1 \text{ if x} \in \mathbb{Q} \\ 0 \text{ if x} \notin \mathbb{Q} \end{cases}$
Let $\{a_1, a_2, ..., a_n\}$ be the set of possible values of f. Let $A_i = \{x : f(x) = a_i\} = f^{-1}(\{a_i\})$. Since f is measurable, $A_i \in \mathcal{B}(\mathbb{R})$.

Define $\int_{-\infty}^{\infty} f(x)dx = \sum_{i=1}^{n} a_i \lambda(A_i)$ provided that there is no $\infty - \infty$ in this sum. Some $\lambda(A_i)$ may be infinity – and this is ok – but we have a problem if two of the coefficients in front of $\lambda(A_i)$ are non-zero. Infinity is just the sequence as $n \to \infty$, but can go to ∞ at different rates.

In particular, if f is a non-negative simple function, then the sum is always well-defined (may be ∞). Then $\int_{-\infty}^{\infty} f(x)dx = 1 \cdot \lambda(\mathbb{R}\backslash\mathbb{Q}) + - \cdot \lambda(\mathbb{Q}) = \infty$

If one of the a_i's is zero, we define the corresponding term $a_i \lambda(A_i)$ to be zero even if $\lambda(A_i) = \infty$.

Ex. $f(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq 1 \\ 0 \text{if } x \notin [0, 1] \end{cases}$.
Then $\int_{-\infty}^{\infty} f(x)dx = 1 \cdot \lambda([0, 1]) + 0 \cdot \lambda(\mathbb{R}\backslash[0, 1]) = 1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function. Let $SF^+(\text{f}) = \{g : \text{g is a non-negative simple function and } g(x) \leq f(x) \forall x\}$.

Define: $\int_{-\infty}^{\infty} f(x)dx = \sup\{\int_{-\infty}^{\infty} g(x)dx | g \in SF^+(f)\}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be any measurable function. Let $f^+(x) = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ 0 \text{ if } f(x) < 0 \end{cases}$

Similarly, define $f^-(x) = \begin{cases} -f(x) \text{ if } f(x) \leq 0 \\ 0 \text{ if } f(x) > 0 \end{cases}$.

Observe $f = f^+ - f^-$, f^+, f^- are both nonnegative measurable functions. $|f| = f^+ + f^-$.

We know how to compute $\int_{-\infty}^{\infty} f^+(x)dx$ and $\int_{-\infty}^{\infty} f^-(x)dx$. If at least one of these integrals is finite, we define the integral of f as: $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f^+(x)dx - \int_{-\infty}^{\infty} f^-(x)dx$ [so we don't have the problem of $\infty - \infty$]. If both integrals are infinite, we say the Lebesgue integral is not defined.

Ex. $f(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ \frac{\sin x}{x} \text{ if } x > 0 \end{cases}$. Then f^+ and f^- are both ∞. Observe that $\lim_{L\to\infty} \int_0^L \frac{\sin x}{x} dx = \frac{\pi}{2}$, but if you define the area under the curve differently (or depending on how you sum this area), you will get an undefined Lebesgue integral.

26.2 Monotone Convergence and Dominated Convergence

Theorem 26.1 *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded, Riemann integrable function. Then $\exists g : [a, b] \to \mathbb{R}$ which is measurable, equal to f almost everywhere, and the Lebesgue integral of g equals the Riemann integral of f. In particular, if f is measurable, then the Lebesgue integral = the Riemann integral.*

Def.[Lebesgue Integral on an Interval] Given a function $g : [a, b] \to \mathbb{R}$, the Lebesgue integral $\int_a^b g(x)dx$ is defined to be the integral $\int_{-\infty}^{\infty} g(x)dx$ where $g(x) := 0$ if $x \notin [a, b]$ provided that this g is measurable and $\int_{-\infty}^{\infty} g(x)dx$ exists.

Lots of functions that are Lebesgue integrable and not Riemann integrable. Works for any set where you can define a measure on that set, then for any real-valued measurable function, you can define a Lebesgue integrable [not just on the real line]. Can work for $\mathbb{R}^d$, manifolds, or infinite dimensional spaces.

For Riemann integrals, it's hard to exchange limits and integrals (need uniform convergence). For Lebesgue integrals, it's a bit better.

Theorem 26.2 (Monotone convergence Theorem) *Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative measurable functions increasing to a function f. (That is $\forall x \in \mathbb{R}$, $f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots$ and $\liminf f_n(x) = f(x)$). Then $\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx$.*

Theorem 26.3 (Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions converging pointwise to a function f. Suppose that $\exists g : \mathbb{R} \to [0, \infty)$ which is measurable,a nd $\int_{-\infty}^{\infty} g(x)dx < \infty$, and $|f_n(x)| \leq g(x)$, $\forall n, x$. Then $\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx$*

These two theorems are some of the main reasons why Lebesgue integration is very useful. No longer need uniform convergence to swap limits and integrals (limit of integrals is the integral of the limit).

26.3 Fun Exploration [Kolmogorov-Arnold]

$f : [0, 1]^2 \to \mathbb{R}$. $f(x, y) = x + y$. $f(x, y) = xy = exp(\log x + \log y)$.

One of Hilbert's 20 questions: how many questions of multiple variables can be expressed as a sum of functions of other functions. xy is multiplication which is equal to the sum of $exp(\log x + \log y)$. Amazingly, this is true (1957-1956).

Theorem 26.4 (Kolmogorov-Arnold Theorem) *Take any $n \geq 2$, $\exists$ functions $\phi_{pq} \in C[0,1]$, $p = 1, ..., n$, $q = 1, ..., 2n$ such that any $f \in C([0,1]^n)$ [space of continuous functions of n-variables to be restricted between 0 and 1] can be represented as $f(x_1, ..., x_n) = \sum_{q=1}^{2} n g_q(\sum_{p=1}^{n} \phi_{pq}(x_p))$ for some $q_1, ..., q_{2n} \in C[0,1]$ depending on f. Note ϕ_{pq} does not depend on f!! ϕ_{pq} is universal. There are no multi-variate functions – it's all just a sum of univariate functions. But q's do.*

This gave rise to the field of approximation theory (relevant to neural networks).

Proof:

Theorem 26.5 (Baire Category Theorem – Simplest Version) *Let (x, d) be a complete metric space. Let $D_1, D_2, D_3, ...$ be a sequence of open, dense subsets of X. Then $\cap_{n=1}^{\infty} D_n$ is dense (and in particular is non-empty).*

Def.[Dense] A set $A \subseteq X$ if $\overline{A} = X$. For example, the rational numbers are dense in the real line (since the closure of the rational numbers equals the reals).

Rationals and irrationals are both dense in $\mathbb{R}$, but their intersection is empty. In this case, if we have a sequence of open dense subsets, then the intersection is dense (and therefore non-empty).

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27.1 Kolmogorov-Arnold Theorem

Theorem 27.1 (Baire Category Theorem) *Let (X, d) be a complete metric space. Let $D_1, D_2, ...$ be open, dense subsets of X. Then $\cap_{i=1}^{n} D_i$ is dense.*

Proof:
Take any $x \in X$ and $\epsilon > 0$.
To show $\exists y \in \cap_{i=1}^{n} D_i$, such that $y \in \beta_\epsilon(x)$ a subset is dense if and only if every open ball has an element of that set.

Since D_1 is dense, $\exists y_1 \in D_1 \cup \beta_{\frac{\epsilon}{2}}(x)$. Since D_1 is open, $\exists r$ such that $\beta_r(y_1) \subseteq D_1$. Take $r_1 = \min\{r, \frac{\epsilon}{2}\}$. Then $\beta_{r_1}(y_1) \subseteq D_1$ and $\beta_{r_1}(y_1) \subseteq \beta_\epsilon(x)$.

Since D_2 is dense, $\exists y_2 \in D_2 \cap \beta_{r_{\frac{1}{2}}}(y_1)$. Again, since D_2 is open, $\exists r > 0$ such that $\beta_r(y_2) \subseteq D_2$. Let $r_2 = \min\{r, \frac{r_1}{2}\}$. Then $\beta_{r_2}(y_2) \subseteq D_2$. $\beta_{r_2}(y_2) \subseteq \beta_{r_1}(y_1) \subseteq$

$\beta_\epsilon(x)$. [Correction: choose $r_{i+1} \leq \frac{r_i}{4}$ instead of $\frac{r_i}{2}$ – ensures the limit cannot converge to the boundary of the open ball.]

We proceed in this way to find $y_1, y_2, y_3, ...$ and $r_1, r_2, r_3, ...$ such that $\forall i$, $\beta_{r_i}(y_i) \subseteq D_i$.
(2) $\forall i$, $\beta_{r_{i+1}}(y_{i+1}) \subseteq \beta_{r_i}(y_i)$
(3) $r_{i+1} \leq \frac{r_i}{2}\ \forall i$
(4) $\beta_{r_1}(y_1) \subseteq \beta_\epsilon(x)$.

Then note that $\forall i < j$, $y_j \in \beta_{r_j}(y_j) \subseteq \beta_{r_{j-1}}(y_{j-1}) \subseteq ... \subseteq \beta_{r_j}(y_i) \Rightarrow d(y_i, y_j) < r_i \leq \frac{\epsilon}{2^i}$.

Thus, $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $y = \lim\limits_{n\to\infty} y_n$ which exists since (X, d) is a complete metric space. Note that, for each i, $d(y_i, y) = \lim\limits_{j\to\infty} d(y_i, y_j) \leq r_i$. [But this isn't good enough – we want strictly less than]. Thus, for any i, $d(y_i, y) \leq d(y_i, y_{i+1}) + d(y_{i+1}, y)$.
$d(y_i, y) \leq d(y_i, y_{i+1}) + d(y_{i+1}, y) \leq \frac{r_i}{2} + \frac{r_i}{4} = \frac{3r_i}{4} < r_i$.

Find r such that $\beta_r(y_{i+1}) \subseteq D_{i+1}$. Let $r_{i+1} = \min\{r, \frac{r_i}{4}\}$. Therefore, $y \in \beta_{r_i}(y_i) \subseteq D_i\ \forall i \Rightarrow y \in \cap D_i \Rightarrow y \in \beta_\epsilon(x)$.

Proof-sketch for Kolmogorov-Arnold Theorem
Take any n. Let $\Phi = \{\phi \in C[0,1] | \phi \text{ is increasing. }, \phi(0) = 0, \phi(1) = 1\}$.

Fix some $\lambda_1, ..., \lambda_n > 0$ which are distinct and $\sum_{i=1}^n \lambda_i = 1$. Choose some $\epsilon > 0$ (to be determined later). For any $f \in C([0,1]^n)$, $f \neq 0$. Define $\omega(f)$ to be the set of all $(\phi_1, ..., \phi_{2n+1}) \in \Phi^{2n+1}$ such that for some $h \in C[0,1]$, we have $||h|| \leq ||f||$ where $||f|| = \sup_{x\in[0,1]^n} |f(x)|$, and $\forall x_1, x_2, ...x_n \in [0,1]$, $|f(x_1, ..., x_n) - \sum_{q=1}^{2n+1} h(\sum_{p=1}^n \lambda_p \phi_q(x_p))| < (1-\epsilon)||f||$ [this is an approximation of f].

Endow Φ^{2n+1} with the metric $d((\phi_1, ..., \phi_{2n+1}), (\psi_1, ..., \psi_{2n+1})) = \sum\limits_{i=1}^{2n+1} ||\phi_i - \psi_j||$ where $||.||$ is the supremum metric.

Clearly, $\omega(f)$ is open. A complicated construction shows that for a small enough choice of ϵ, $\omega(f)$ is dense in $\Phi^{2n+1}\ \forall f$.

Fact: $C([0,1]^n)$ has a countable dense subset. Let's say F. By the Baire Category Theorem, $\cup_{f\in F}\Omega(f)$ is nonempty. Take any $(\phi_1, ..., \phi_{2n+1}) \in \cap_{f\in F}\Omega(f)$. Take any $f_0 \in C([0,1]^n)$. Take $f \in F$ such that $||f|| \leq ||f_0||$, $||f - f_0|| < \frac{\epsilon}{2}||f_0||$. Since $(\phi_1, ..., \phi_{2n+1}) \in \cup_{g\in F}\Omega(g) \subseteq \Omega(f)$, there exists $h_0 \in C[0,1]$ such that $||h_0|| \leq ||f||$ and $|f(x_1, ..., x_n) - \sum_{q=1}^{2n+1} h_0(\sum_{p=1}^n \lambda_p \phi_q(x_p))| < (1-\epsilon)||f||$.
$\Rightarrow |f_0(x_1, ..., x_n) - \sum_{q=1}^{2n+1} h_0(\sum_{p=1}^n \lambda_p \phi_q(x_p))| < (1-\epsilon)||f_0||$

Then take the remainder $f_1(x_1, ..., x_n)$ and find h_1, etc. Continue this process:

$f_0(x_1, ..., x_n) = \sum_{q=1}^{2n+1} \sum_{j=0}^{\infty} h_j(\sum_{p=1}^{n} \lambda_p \phi_q(x_p))$

It follows that $||h_j|| \to 0$ exponentially fast. $h = \sum_{j=1}^{\infty} h_j \in C[0, 1]$.